Asymptotic limit analysis for compatible numerics in numerical weather prediction

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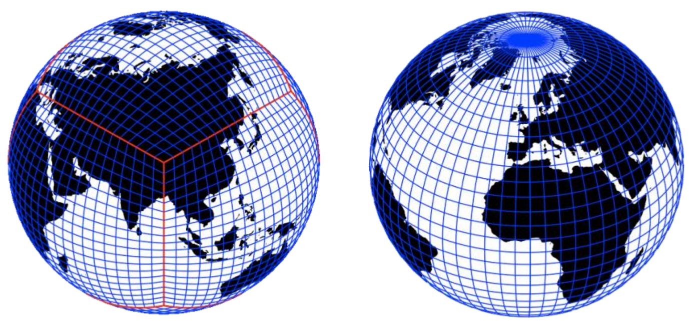
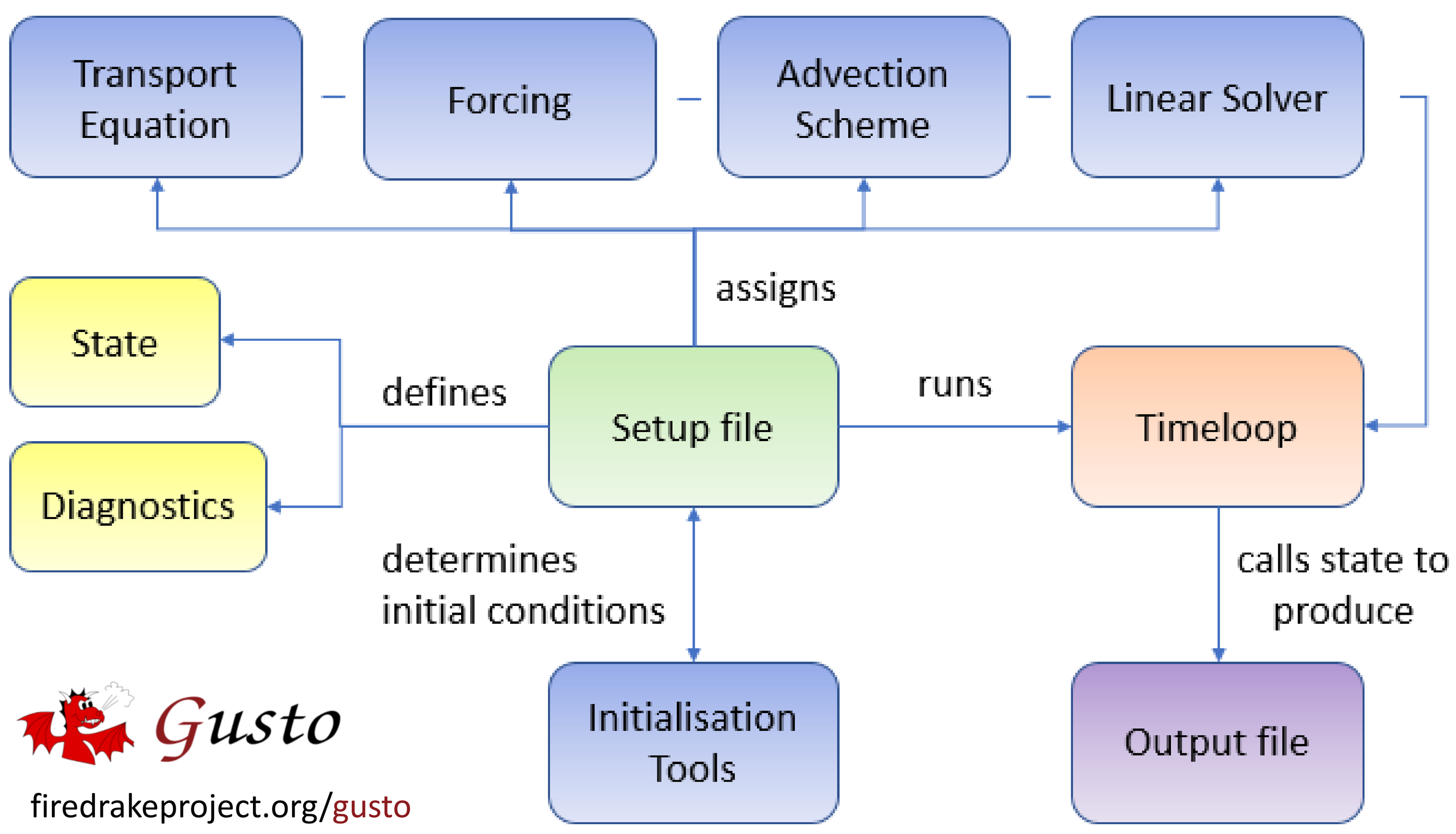
Introduction

To avoid parallel-computing issues associated with longitude-latitude grids in numerical weather prediction, the UK Met office is developing a dynamical core called Gung Ho, which is based on the compatible finite element approach, thus allowing the use of pseudo-uniform grids on the sphere. One way to examine this approach is via an asymptotic limit analysis, where the model's behaviour approaching a parameter limit is considered.



Gusto

For the numerical modelling part of this project, we use and expand Gusto, a dynamical core library using compatible finite element methods developed in the Gung Ho project. Gusto is based on Firedrake, a freely available finite element code generation library, and is maintained and developed by members of the mathematics department at Imperial College London. It is build in a modular design, allowing for a simple combination of numerical schemes and underlying dynamical equations.



Vertical Slice Model

We consider as a test case vertical wind perturbations induced by a steady horizontal flow over a small mountain. The underlying dry Euler equations are given by

$$\delta_V\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + c_p\theta\nabla\pi = -g\mathbf{k}$$ Momentum equation for wind velocity $\mathbf{u}$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$ Mass equation for density ρ

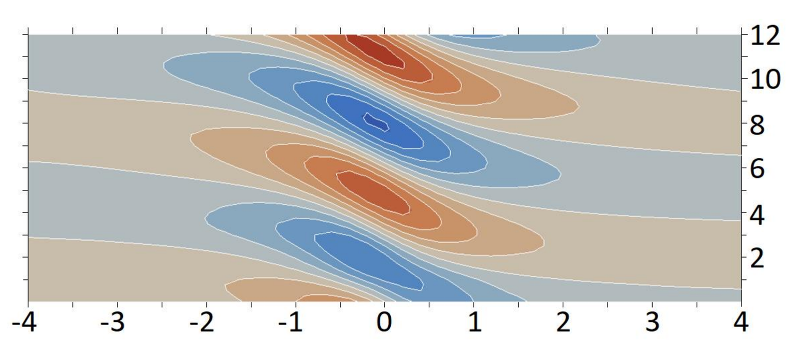
$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = 0$$ Thermodynamic equation for pot. temperature θ

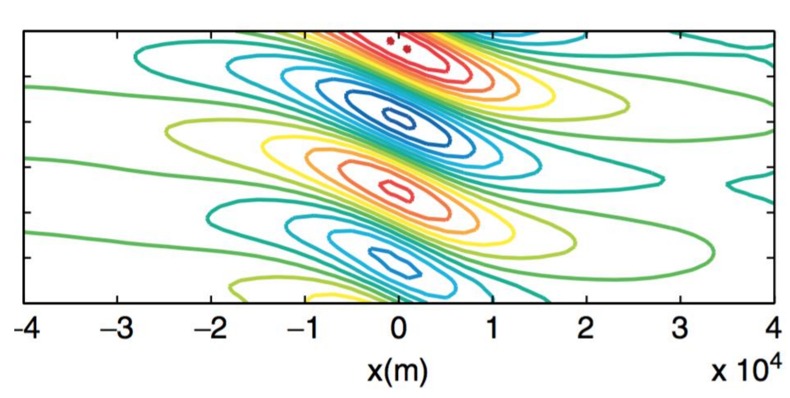
$$\pi^{\frac{1-\kappa}{\kappa}} = \frac{R}{p_0}\rho\theta$$ Ideal gas law for Exner pressure π

Other terms: specific heat c_p, gravitational force g, vertical unit vector $\mathbf{k}$, gas constant R, $\kappa = {}^{R}/_{c_p}$, reference pressure p_0

δ_V acts as a switch between the non-hydrostatic and hydrostatic setups, setting the vertical wind acceleration zero in the latter case. The spatial discretisation is based on the compatible finite element method, using the so-called Euler-Poincaré form of the velocity advection term. For stability, we use an upwind DG formulation in the mass and momentum equations, and a mix of upwind DG and SUPG for the temperature equation. A semi-implicit scheme is used for the time discretisation, with separated advection and forcing steps. The resulting non-linear system is solved using a linearised Newton method in residual form.

The resulting vertical wind perturbation patterns for the hydrostatic case are depicted on the left, with contour lines every 5×10^{-4} m/s, for horizontal wind speed 20m/s, hill height 1m. For comparison, the right-hand side image depicts the corresponding result from the current Met office dynamical core [7].





Compatible Finite Element Method

The use of different finite element spaces for a system of PDEs for different unknowns leads to a mixed finite element discretisation. For stability, the relationship between the chosen subspaces needs to be kept in mind. On simplicial cells (e.g. rectangles), a stable discretisation can be ensured by choosing 'compatible' spaces, i.e. spaces that are mapped to each other by differential operators, ensuring that vector calculus identities hold in the discretisation.

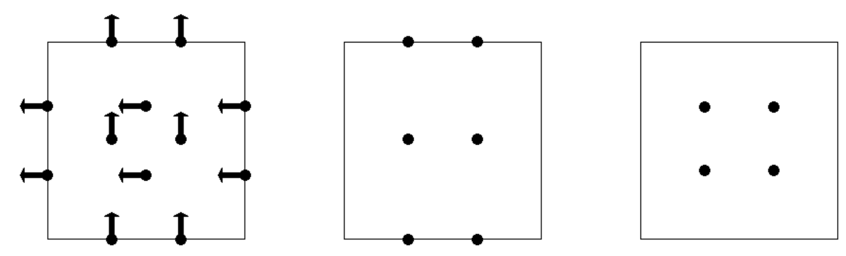
In 1D, we would pick finite element velocity and pressure spaces V_0, V_1 such that

$$V_0 \xrightarrow{\frac{d}{dx}} V_1$$

meaning that differentiation maps the velocity finite element space onto the pressure space. This can be extended to the 2D case via the following construction:

$$W_0 \xrightarrow{\nabla\cdot} W_1 : \qquad \begin{aligned} W_0 &= \mathrm{HDiv}(V_0\otimes V_1) \oplus \mathrm{HDiv}(V_1\otimes V_0) \\ W_1 &= V_1\otimes V_1 \end{aligned}$$

For a vertical slice, an example of the resulting rectangular elements for velocity, temperature and pressure is depicted below, where black dots and arrows represent degrees of freedom and multiplication by the corresponding unit vector respectively.

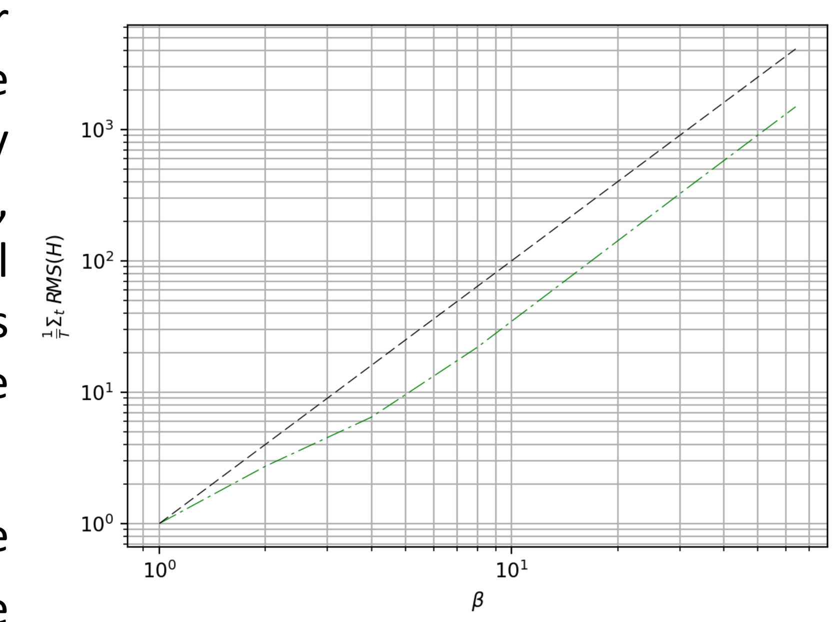


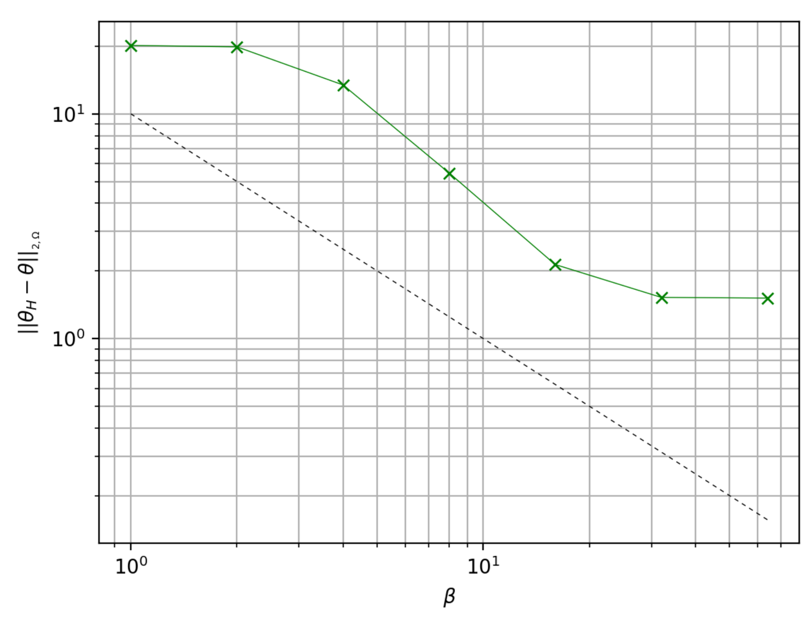
Asymptotic Limit Analysis

A non-dimensionalisation of the dry Euler equations shows that, at constant Froude number, hydrostaticity is determined by the domain's length to height ratio β. Thus, we expect the non-hydrostatic model results to converge to the hydrostatic ones as we stretch the domain. To verify this, we consider two measures:

- $\frac{1}{T}\sum_t RMS(H)$, a time-average over the normalised L2-norm of the hydrostaticity in the slice
- $\| \theta_H - \theta \|_{2,\Omega}$, the L2-norm difference between the temperature perturbation in hydrostatic and non-hydrostatic setup

We expect the mean hydrostaticity to grow in second order and the difference in temperature to converge to zero as beta increases. The numerical results confirm this, albeit with the temperature difference staying level from a certain point on due to a soundwave-reducing sponge layer inserted near the top of the domain.





Future Work

A desired feature of dynamical cores for weather and climate prediction is conservation of various entities of the system, such as energy and mass. One way to achieve this is by considering the discretisation in a Hamiltonian/variational framework. We are currently exploring ways to implement this framework in Gusto, while keeping the upwind schemes used to enhance stability.

References

[1] MJP Cullen. Modelling atmospheric flows. Acta Numerica, 16:67-154, 2007.

[2] R Klein. Asymptotic analyses for atmospheric flows and the construction of asymptotically adaptive numerical methods. ZAMM Zeitschrift für angewandte Mathematik und Mechanik, 80(11-12):765-785, 2000.

[3] A Natale, CJ Cotter and J Shipton. Compatible finite element spaces for geophysical fluid dynamics. Dynamics and Statistics of the Climate System, 1(1).

[4] A Natale and CJ Cotter. A variational h (div) finite element discretisation for perfect incompressible fluids. IMA J. Numer. Anal. 2017.

[5] AR Visram, CJ Cotter, and MJP Cullen. A framework for evaluating model error using asymptotic convergence in the eady model. Quarterly Journal of the Royal Meteorological Society, 140(682):1629-1639, 2014.

[6] R Ford et al. Gung Ho: A code design for weather and climate prediction on exascale machines. In Proceedings of the Exascale Applications and Software Conference, 2013.

[7] T Melvin et al. An inherently mass-conserving iterative semi-implicit semi-Lagrangian discretization of the non-hydrostatic vertical-slice equations. Quarterly Journal of the Royal Meteorological Society, 136(648):799-814, 2010.

Sphere grid images: MCORE, University of Michigan

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