Discretising the Stochastic Quasi-Geostrophic Equations

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Motivation

Fluids have motions over a vast range of scales, and so most numerical models will not resolve some of these scales. Parametrisation is the process of attempting to describe the effects of the unresolved flow upon the resolved flow. Making parametrisations stochastic allows models to explore more of their phase space. However, an arbitrary stochastic parametrisation may not preserve properties of the fluid.

To this extent, Holm proposed a fluid formulation in [1] whereby the velocity has a stochastic component. The formulation is constructed via a variational principle in such a way that the circulation theorem is preserved by the stochastic velocity.

As part of [1], Holm gives a stochastic version of the quasi-geostrophic (QG) equations. The aim of this project was to explore the stochastic QG system via numerical simulation. A major part of this was to present a numerical scheme for the system and to investigate its statistical properties.

Quasi-Geostrophic Theory

Quasi-geostrophic (QG) theory approximates the motion of shallow geophysical fluids for which the Coriolis force and the pressure gradient force are nearly in balance. The theory applies to scales of flow with small Rossby number: rotational forces are much larger than inertial forces.

The system is two-dimensional and described by one prognostic variable: the potential vorticity Q, which is defined by

$$Q := \nabla^2\psi - \mathcal{F}\psi + f, \tag{1}$$

where f is the Coriolis parameter, so gives the potential vorticity contribution from the Earth's rotation and $\mathcal{F}$ is the reciprocal of the Burger number, so describes the scale of motion of the fluid. The stream function ψ is related to the fluid velocity $\boldsymbol{v}$ by $\boldsymbol{v} = \boldsymbol{\nabla}^\perp\psi$, where $\boldsymbol{\nabla}^\perp := (-\partial_y, \partial_x)$.

The governing equation of the QG system describes the conservation of Q under advection by $\boldsymbol{v}$, so that

$$\frac{\partial Q}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}Q = 0. \tag{2}$$

For a detailed description, see [2].

Rossby Waves on the Unit Square

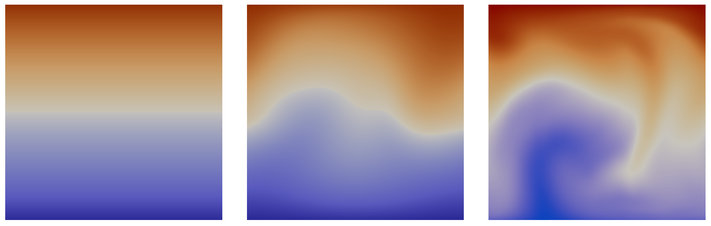

Figure 1: The generation of a Rossby wave on the unit square by perturbing the potential vorticity profile $Q = Uy$ and using periodic boundary conditions on $x = 0$ and $x = 1$.

The Stochastic Quasi-Geostrophic Equations

We derived the form of these equations by substituting the quasi-geostrophic (QG) Lagrangian from [3] into Holm's variational formulation in [1]. This gave

$$Q := \nabla^2\psi - \mathcal{F}\psi + f, \tag{3}$$

$$\mathrm{d}Q + \boldsymbol{\nabla}Q\cdot\boldsymbol{\nabla}^\perp\left(\psi\,\mathrm{d}t + \sum_i \xi_i(\boldsymbol{x})\circ\mathrm{d}W_i\right) = 0, \tag{4}$$

where the parameters $\mathcal{F}$ and f and the operator $\boldsymbol{\nabla}^\perp$ are defined in the Quasi-Geostrophic Theory section. The $\xi_i(\boldsymbol{x})$ functions are stream functions giving rise to the stochastic part of the velocity, whilst the $\mathrm{d}W_i$ are Wiener processes and the $\circ$ symbol denote that the noise obeys Stratonovich calculus.

Simulations on the Sphere

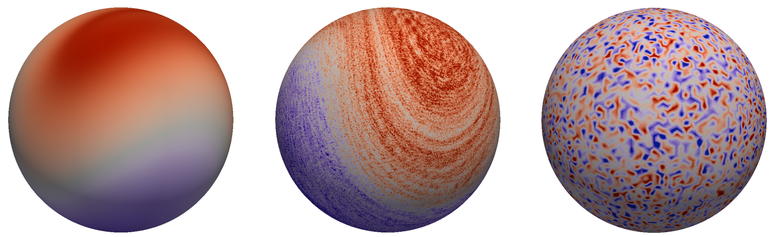

Figure 2: The simulation of the stochastic QG equations on the surface of a sphere.

An Enstrophy-Conserving Numerical Scheme

The general strategy behind the numerical scheme can be broken into three steps:

1. Find ψ from Q by inverting the definition of Q:
$$\psi = \left(\nabla^2 - \mathcal{F}\right)^{-1}(Q - f). \tag{5}$$
2. Determine the stochastic contributions to the velocity by randomly generating the $\mathrm{d}W_i$ for that time step.
3. Compute the value of Q at the next time step from the stochastic velocity and the previous value of Q.

Multiplying equations (3) and (4) by test functions ϕ, γ respectively and then integrating gives

$$\int_\Omega (\mathcal{F}\phi\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi)\,\mathrm{d}^2x = \int_\Omega \phi(f - Q)\mathrm{d}^2x, \tag{6}$$

$$\mathrm{d}\int_\Omega \gamma Q\,\mathrm{d}^2x - \int_\Omega \boldsymbol{\nabla}\gamma\cdot\boldsymbol{\nabla}^\perp\left(\psi\,\mathrm{d}t + \sum_i \xi_i(\boldsymbol{x})\circ\mathrm{d}W_i\right) Q\,\mathrm{d}^2x. \tag{7}$$

A finite element discretisation is introduced by choosing some continuous finite element space V, and defining $\mathring{V}$ as

$$\mathring{V} = \{\psi \in V : \psi = 0 \text{ on } \partial\Omega\}. \tag{8}$$

The discretisation is found by choosing $(Q, \psi) \in (V, \mathring{V})$ so that equations (6) and (7) are true for all $(\gamma, \phi) \in (V, \mathring{V})$. The time discretisation uses an implicit midpoint rule.

Statistical Properties of the Numerical Scheme

The numerical scheme conserves the two integrals over the domain Ω,

$$\Pi := \int_\Omega Q\,\mathrm{d}^2x \quad \text{and} \quad Z := \tfrac{1}{2}\int_\Omega Q^2\,\mathrm{d}^2x. \tag{9}$$

This can be shown by substituting $\gamma = 1$ and $\gamma = Q$ into equation (7).

In a finite element discretisation, the field Q is written as the sum of N basis functions, $\phi_i(\boldsymbol{x})$, so that $Q = \sum_i^N Q_i\phi_i(\boldsymbol{x})$. The state of the system $\boldsymbol{Q}$ is stored as a point in $\mathbb{R}^N$ by the collection of coefficients Q_i. Following the work of [4], it is possible to use the maximum entropy principle (as described in [5]) to find that the probability distribution for $\boldsymbol{Q}$ is given by

$$\mathcal{G}(\boldsymbol{Q}) = C\exp\left\{-\alpha\left(Z(\boldsymbol{Q}) + \mu\Pi(\boldsymbol{Q})\right)\right\}, \tag{10}$$

where C is a normalising constant, whilst α and μ provide the constraints that Π and Z are conserved.

If the stochastic system is ergodic, then the trajectory through phase space of a long simulation should mimic this probability distribution. This was tested using the Metropolis algorithm to generate samples from $\mathcal{G}(\boldsymbol{Q})$. The average quantities from these samples were compared with the time-averaged quantities measured from a long simulation.

This experiment was performed on a spherical domain, and showed that the samples taken from the Metropolis algorithm replicated the properties of the fluid simulation.

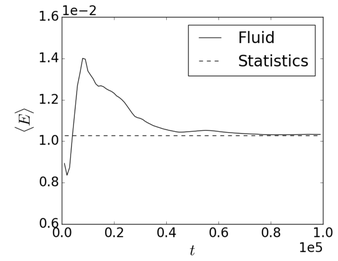

Figure 3: The average energy over time of a stochastic fluid simulation converged to the energy predicted by the Metropolis algorithm.

References

[1] D. D. Holm, "Variational principles for stochastic fluid dynamics," in *Proc. R. Soc. A*, vol. 471, p. 20140963, The Royal Society, 2015.

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[4] S. Dubinkina and J. Frank, "Statistical mechanics of Arakawa's discretizations," *Journal of Computational Physics*, vol. 227, no. 2, pp. 1286–1305, 2007.

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