Stochastic Filtering for Rotating Shallow Water Equations

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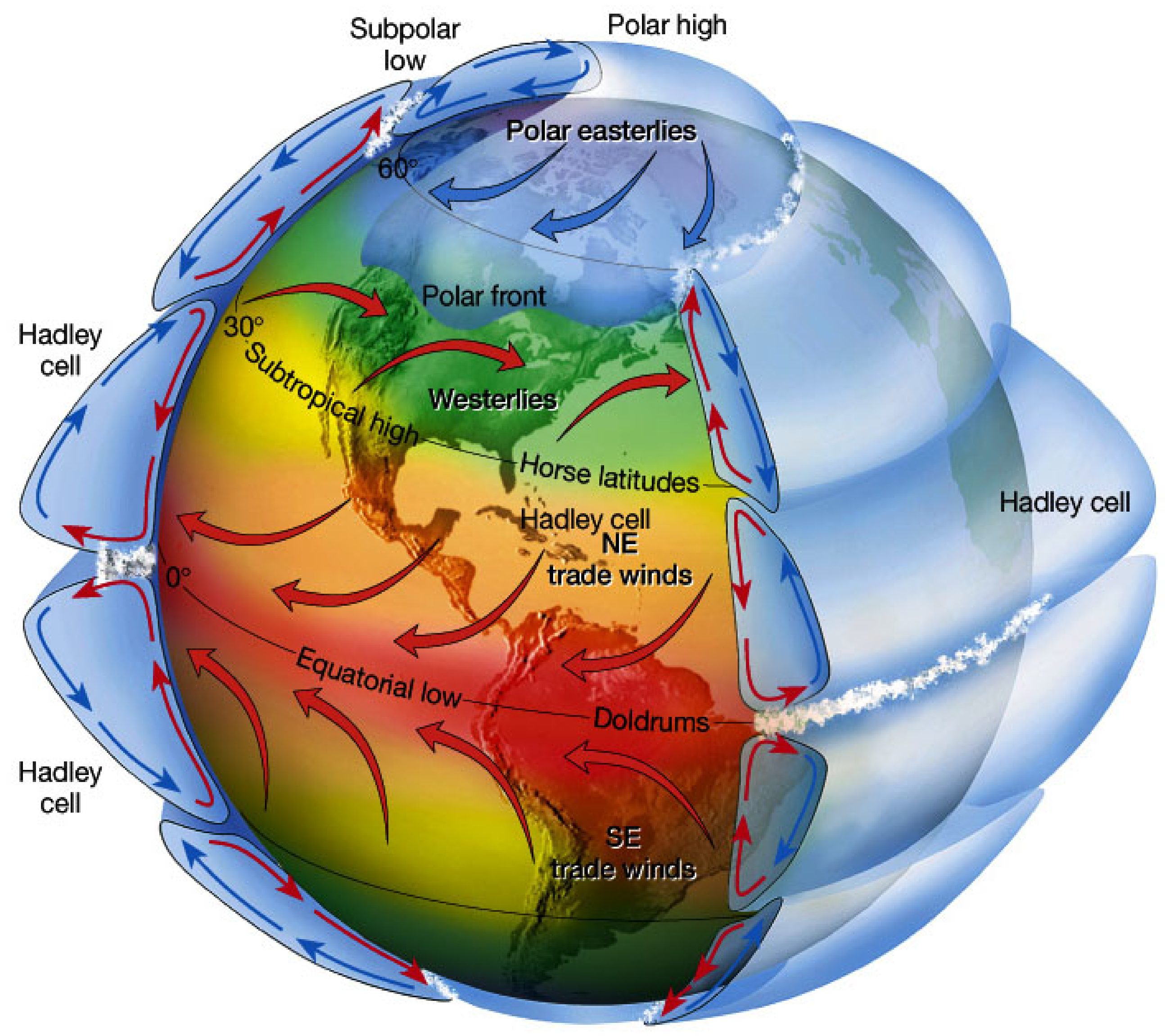
Introduction

We investigate a **stochastic filtering/data assimilation problem** consisting of a signal which models the motion of an incompressible fluid below a free surface when the vertical length scale is much smaller than the horizontal one. The evolution of the two-dimensional rotating system is represented by an infinite dimensional stochastic PDE and observed via a finite dimensional observation process. The deterministic part of the SPDE consists of **a classical shallow water equation**, while the stochastic part involves a **transport type noise**.

Motivation

Climate change is one of the most challenging problems that we are currently facing, fundamental research in this area being, therefore, of crucial importance. Large-scale circulatory motions of ocean and atmosphere involve complex geophysical phenomena which have a determinant influence on climate dynamics. Although one will never be able to formalise reality in its entire complexity, the **introduction of stochasticity into ideal fluid dynamics** is a realistic tool for modelling the sub-grid scale processes that cannot otherwise be resolved.

Atmospheric Circulation - a central topic in Data Assimilation



Source: http://www.ux1.eiu.edu/ cfjps/1400/circulation.html

Introducing stochasticity...

Following [2] we introduce the noise:

$$\sum_{i=1}^{\infty}[\xi_i, q] \circ \mathrm{d}B_t^i,$$

where q is the advected quantity, ξ_i are divergence free vector fields corresponding to the underlying physics, B_t^i are independent Brownian motions, and $[\cdot,\cdot]$ is the Jacobi-Lie bracket.

Why is this the 'right' noise?

The hierarchy of model equations which govern climate dynamics is designed to provide the most accurate approximation for the original equations near a geostrophically balanced state. One of the necessary conditions required in this hierarchy is the existence of a conservation law. By adding this type of noise to the deterministic part, certain **physical properties of the solution of the resulting stochastic PDE are conserved**.

Rotating Shallow Water Equations and the Barotropic Vorticity Model

The dynamics of **a rotating shallow water system** on a two dimensional domain is given, in nondimensional form, by

$$\epsilon\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} + \nabla p = 0 \qquad \text{and} \qquad \frac{\mathrm{d}h}{\mathrm{d}t} + \nabla\cdot(h\mathbf{u}) = 0,$$

where: ϵ is the Rossby number, $\mathbf{u}$ is the horizontal fluid velocity vector, f is the Coriolis parameter, $\hat{\mathbf{z}}$ is a unit vector pointing away from the center of the Earth, h is the thickness of the fluid, $p := \frac{h}{\epsilon\mathcal{F}}$ and $\mathcal{F}$ is the Froude number.

These two equations are classically derived from the momentum and mass continuity equations which fully characterise the motion of a fluid.

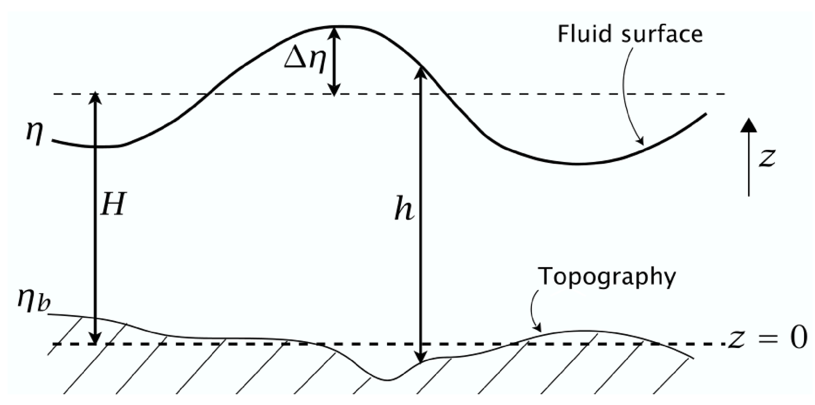


Figure : A single layer shallow water system. η is the total depth, η_b is the height of the bottom surface, $h := \eta - \eta_b$, and H is the mean depth.

An important feature of a turbulent flow is the rotation of the velocity field. For this reason, a useful model in the study of atmospheric and oceanic turbulence is the **vorticity equation**:

$$\mathrm{d}q + (\mathbf{u}\cdot\nabla q)\mathrm{d}t = 0,$$

where $q = \omega/h$ is the potential vorticity, $\omega = \hat{\mathbf{z}}\cdot curl\,(\epsilon\mathbf{u} + \mathbf{R}(x))$ is the total vorticity, and $\mathbf{R}(x)$ is the vector potential of the zero divergence rotation rate about the vertical direction.

In **data assimilation** the **barotropic vorticity equation** is used to model the behaviour of an incompressible, non-viscous, barotropic flow:

$$\mathrm{d}\omega + (\mathbf{u}\cdot\nabla\omega)\mathrm{d}t = 0,$$

where $\omega = curl\mathbf{u}$. Despite the fact that it does not allow for gravity waves, this is a suitable model for highly chaotic regimes, the solutions of the equation above being vortices which strongly interact, grow and dissipate.

Figure from G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics* [4]

Filtering for Stochastic Rotating Shallow Water Equations

Our research is focused on the analysis of filtering problems where the **signal** satisfies the following SPDE:

$$\mathrm{d}q + [\mathbf{u}, q]\mathrm{d}t + \sum_{i=1}^{\infty}[\xi_i, q]\circ\mathrm{d}B_t^i = 0 \qquad \text{and} \qquad \mathrm{d}h + \nabla\cdot(h\mathbf{u})\mathrm{d}t + \sum_{i=1}^{\infty}(\nabla\cdot(\xi_i h))\circ\mathrm{d}B_t^i = 0.$$

The qualitative study of this stochastic system concerns issues like existence, uniqueness, and smoothness properties of the solution. The evolution of the signal will be conditioned by **observations** of an underlying true system, which are pointwise measurements corresponding to the velocity field. According to [1], the resulting conditional distribution (the **posterior distribution**) satisfies the Kushner-Stratonovitch equation. Unfortunately, in most of the cases the corresponding solution cannot be computed explicitly. For this reason, the quantitative analysis will be done using **Ensemble Methods (Particle Filters)**.

Conclusion

A proper **incorporation of stochastics into fluid dynamics** could contribute to the unresolved processes issue. Although this is a single layer model which does not completely reflect the complex stratification of the real atmosphere, it allows for important geophysical phenomena such as gravity and Rossby waves, eddy formation and geophysical turbulence. Consequently, **the models used in data assimilation could be improved significantly**, leading to more precision in weather prediction and climate change issues.

References

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[6] P. J. van Leeuwen, *Aspects of Particle Filtering in High-Dimensional Spaces*, 2015