Accounting for correlated observation errors in variational data assimilation

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What is data assimilation and why are correlated observation errors important?

Motivation:

- Data assimilation combines information from a background field with observations to find the most likely initial state, or analysis.
- Error statistics weight the contribution of observations and background components. They are difficult to estimate, but using incorrect errors negatively impacts the analysis.
- Using correlated observation errors allow high-resolution observations to be exploited fully, with no need for thinning of roughly 80% of observations.
- ECMWF, the Met Office and NRL have all begun to introduce correlated observation error.

Aims: We consider the theoretical implications of using correlated observation errors:

- How is the sensitivity of the analysis affected?
- What is the effect on the amount of computational work required to find the analysis?

Variational assimilation and 3DVar

This work focuses on variational data assimilation - a method where the analysis is found by minimising a cost function. For 3DVar this is given by:

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) + \frac{1}{2}(\mathbf{y}-h[\mathbf{x}])^T\mathbf{R}^{-1}(\mathbf{y}-h[\mathbf{x}]). \quad (1)$$

where the first term keeps the analysis close to the background field and the second term minimises the distance between observations and the analysis in observation space.

The Hessian (matrix of second derivatives) of (1) is given by:

$$\mathbf{S} = \mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}. \quad (2)$$

Sensitivity of the solution of (1) to small changes is investigated by considering the *condition number* of (2). The *condition number* is defined as:

$$\kappa(\mathbf{S}) = \frac{\lambda_{min}(\mathbf{S})}{\lambda_{max}(\mathbf{S})} \quad (3)$$

where $\lambda_{min}(\mathbf{S})$ and $\lambda_{max}(\mathbf{S})$ denote the minimum and maximum eigenvalues of $\mathbf{S}$ respectively.

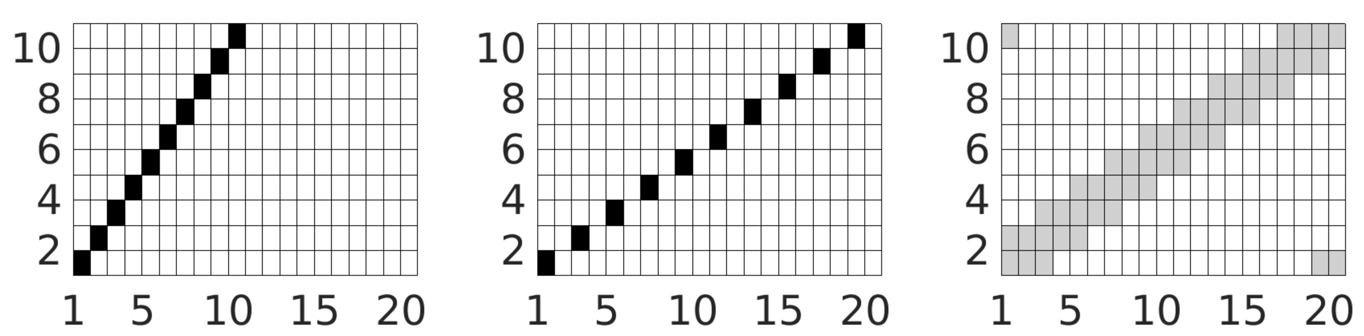
Theorem: New bound on the condition number of the Hessian

Let $\mathbf{B} \in \mathbb{R}^{N\times N}$ and $\mathbf{R} \in \mathbb{R}^{p\times p}$, with $p < N$, be background and observation error covariance matrices. Additionally, let $\mathbf{H} \in \mathbb{R}^{p\times N}$ be the linear observation operator. Then the following bounds are satisfied by the condition number of the Hessian, given by (2):

$$\frac{\kappa(\mathbf{B})}{(1+\frac{\lambda_{max}(\mathbf{B})}{\lambda_{min}(\mathbf{R})}\lambda_{max}(\mathbf{HH}^T))} \le \kappa(\mathbf{S}) \le (1+\frac{\lambda_{min}(\mathbf{B})}{\lambda_{min}(\mathbf{R})}\lambda_{max}(\mathbf{HH}^T))\kappa(\mathbf{B}). \quad (4)$$

Observation operators

The theoretical bound in (4) is tested using three simple choices of observation operator.



(a) $\mathbf{H}_1$ (b) $\mathbf{H}_2$ (c) $\mathbf{H}_3$

Figure 1: Definitions of observation operators used in numerical tests, shown for $p = 10$ observations and $N = 20$ state variables. Shading indicates the value of the entry in the matrix; for $\mathbf{H}_1$ and $\mathbf{H}_2$ all non-zero entries are 1, and for $\mathbf{H}_3$ all non-zero entries are $\frac{1}{5}$.

Importance of observation operator choice

The bounds and actual value of $\kappa(\mathbf{S})$ are calculated for difference choices of $\mathbf{B}$ and $\mathbf{H}$. Known correlation functions were used to generate the correlation matrices $\mathbf{B}$ and $\mathbf{R}$.

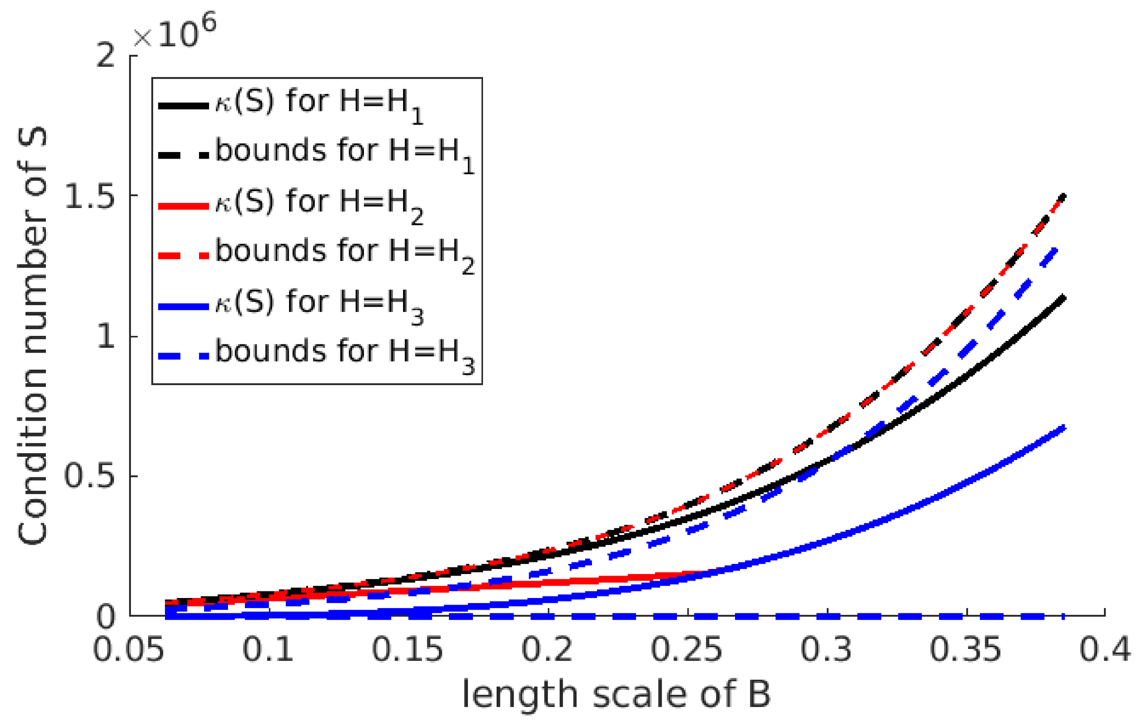


Figure 2: Changing $\kappa(\mathbf{S})$ with the lengthscale of the correlation function of $\mathbf{B}$, for $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ shown in Figure 1 and a fixed choice of $\mathbf{R}$. Dashed lines represent the bounds in (4) and solid lines represent $\kappa(\mathbf{S})$. Different colours represent different choices of observation operator.

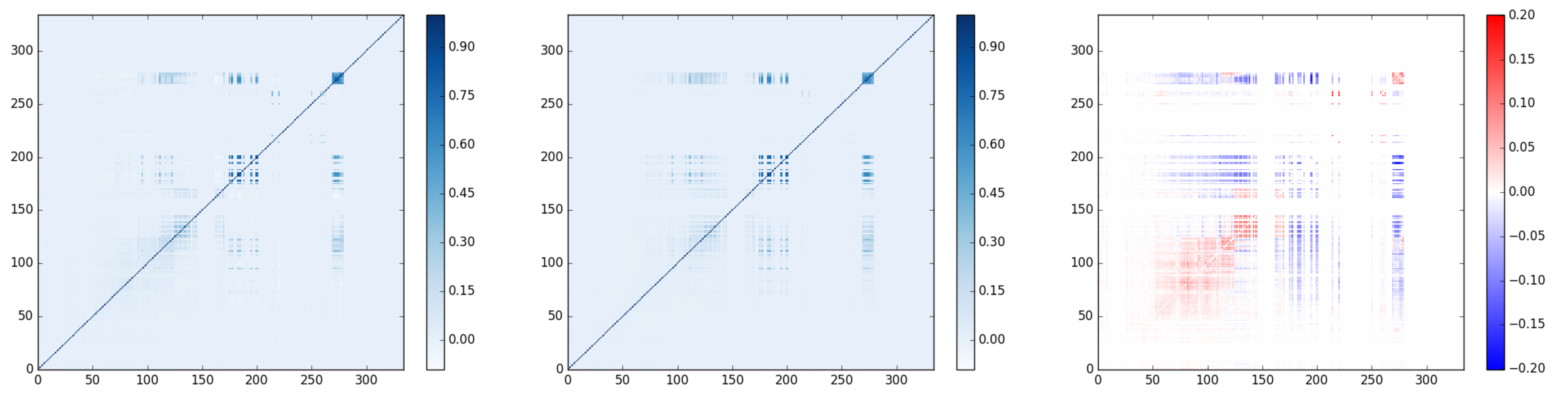
Both $\kappa(\mathbf{S})$ and the upper bound from (4) increase with the lengthscale of $\mathbf{B}$. The choice of observation operator is very important for the value of $\kappa(\mathbf{S})$ - in particular for $\mathbf{H} = \mathbf{H}_1$ and $\mathbf{H} = \mathbf{H}_2$, the bounds are identical, but $\kappa(\mathbf{S})$ is very different.

Key theoretical conclusions

- We have developed a new bound, which numerical tests have shown to be tight.
- A key term for both upper and lower bounds is $\frac{1}{\lambda_{min}(\mathbf{R})}$.
- The choice of observation operator is important
- Both $\mathbf{B}$ and $\mathbf{R}$ have been shown to dominate $\kappa(\mathbf{S})$ for different parameter choices.

Met Office experiments: choices of observation error covariance matrix

- We test our conclusions using the Met Office 1DVar system.
- Observations come from a satellite-based instrument (IASI), so correlations are inter-channel.
- The observation operator is non-linear so the bound of (4) does not apply.
- $\mathbf{R}_{unre}$ had to be 'reconditioned' or altered to improve convergence.
- We investigate how changing $\mathbf{R}$ by increasing its the minimum eigenvalue affects both the condition number of $\mathbf{S}$ and the rate of convergence of 1DVar.



(a) $\mathbf{R}_{unre}$ (b) $\mathbf{R}_{67}$ (c) $\mathbf{R}_{unre} - \mathbf{R}_{67}$

Figure 3: Reconditioning of $\mathbf{R}_{unre}$: 3b is 3a reconditioned so that $\kappa(\mathbf{R}) = 67$ and 3c is the difference between 3a and 3b. Reconditioning causes changes to off-diagonal correlation structure (i.e. correlations between different channels).

Met Office Results: rate of convergence and condition number of Hessian

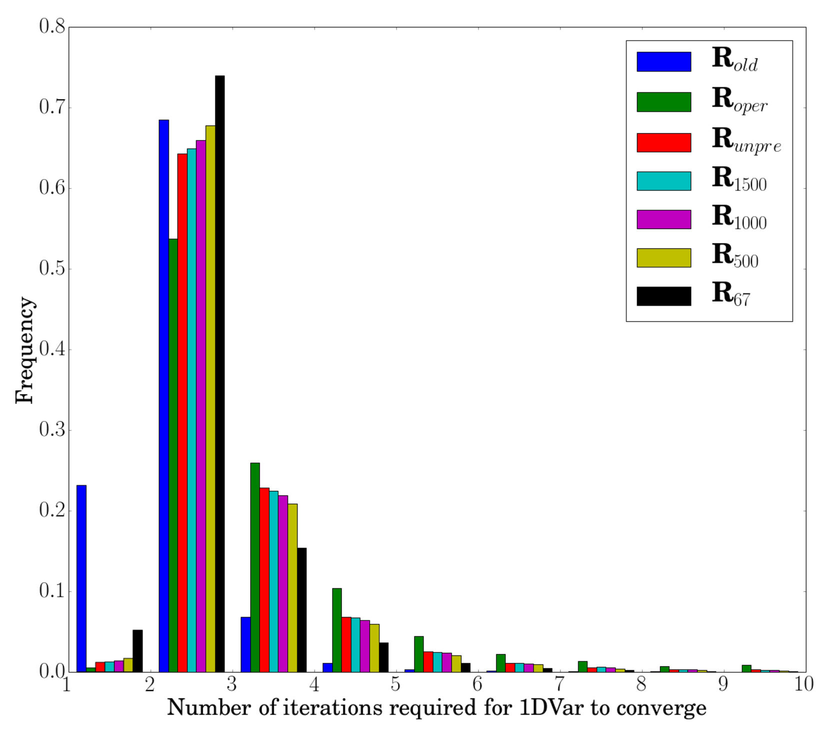


Figure 4: No. of iterations required for convergence for different choices of $\mathbf{R}$.

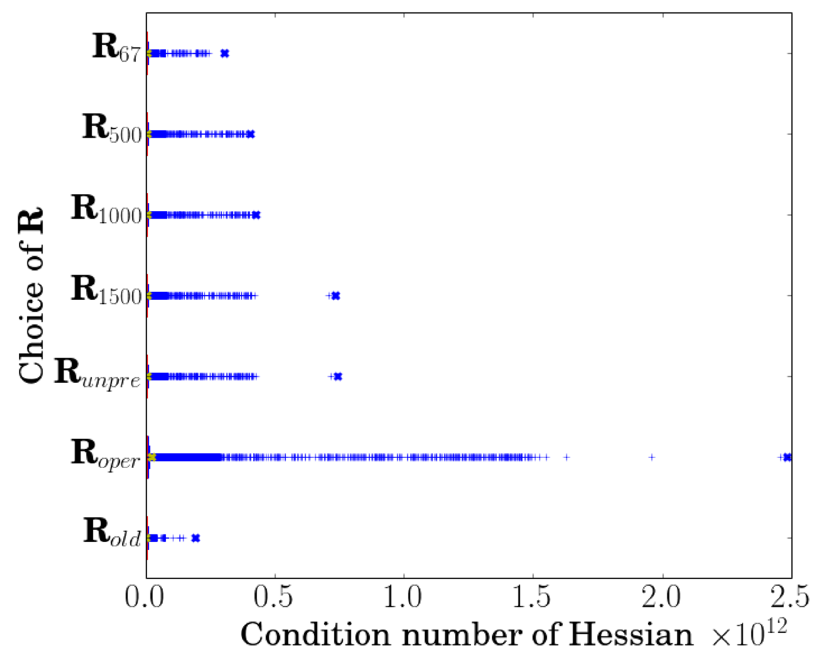


Figure 5: Box plot of condition number of Hessian for each observation for different choices of $\mathbf{R}$.

Figure 5 shows the value of $\kappa(\mathbf{S})$ for each observation, for all choices of $\mathbf{R}$. As $1/\lambda_{min}(\mathbf{R})$ increases, the maximum value of $\kappa(\mathbf{S})$ over all observations decreases.

- Increasing the minimum eigenvalue of $\mathbf{R}$ results in a decrease in $\kappa(\mathbf{S})$ and fewer iterations needed for convergence.
- These conclusions agree qualitatively with theoretical conclusions - in particular with the bound given by (4).

Conclusions

- We have developed a theoretical bound on the condition number of the Hessian in terms of its constituent matrices.
- $1/\lambda_{min}(\mathbf{R})$ is an important term for the condition number of $\mathbf{S}$. Increasing it lowers theoretical bounds, and reduces the condition number in both small scale numerical tests and operational experiments.
- Our choice of observation operator is important for determining the value of $\kappa(\mathbf{S})$. This is true even when the bounds given by (4) are identical.

References

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