Measure and statistical attractors for nonautonomous dynamical systems

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What is a dynamical system?

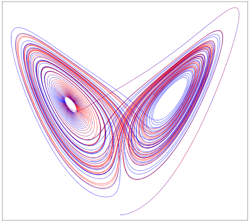
Anything that changes with time is a dynamical system.

- When we know the rule of change, we can attempt to model it!
- What happens in the long run or for different initial points?

Example: Lorenz 63'

$$\begin{cases} \dot{X} = -\alpha X + \alpha Y \\ \dot{Y} = -\alpha X - Y - XZ \\ \dot{Z} = -bZ + XY - b(r + \alpha) \end{cases} \tag{1}$$

- A simplification of a model of the weather
- Varying parameters α, b and r give rise to different behaviour
- the 'rule' or forcing is not changing with time (autonomous)



Autonomous versus nonautonomous

- In dynamical systems theory we mostly focus on *flows*.
- Solutions of ordinary autonomous differential equations give rise to flows,

$$\frac{dx}{dt} = f(x, \lambda),$$

  when f is a Lipschitz function. f does not depend on time.
- What is an nonautonomous dynamical system? The forcing $f(x, t, \lambda(t))$ depends on time explicitly.
- When solutions exist, they also depend on the starting time

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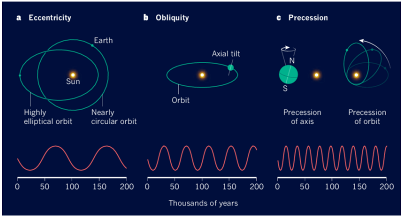
Example: our climate is a nonautonomous dynamical system

The forcing that our climate is subject to are complex, what's more, they change with time:

- Solar radiation: has a daily component of variability and an annual component. Annual can be considered periodic while the daily one, due to cloud cover is better understood as fast varying random component.
- Milankovitch forcing - sort of periodic
- CO2 emissions: increasing trend with small variations



Source: Mark Maslin, Forty years of linking orbits to ice ages, Nature

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There are many notions of attraction for autonomous systems

- Can we say anything about the system in the long term? Does it evolve to a fixed set of values?
- A set S is said ot be invariant if $\phi(t, S) = S$ for all $t \in \mathbb{R}$.
- Important notion in study of dynamical systems: attractor
- There is no universally agreed definition of an attractor! (even for autonomous system)

Definition (classical local attractor)

An invariant compact set $A \subset \mathbb{R}^d$ is called a local attractor of ϕ if there exists an $\eta > 0$ such that

$$\lim_{t \to \infty} d(\phi(t, B_\eta(A)), A) = 0.$$

$d(\cdot, \cdot)$ is the Hausdorff semi-distance.

Classical definition is somewhat restrictive

- the classical definition requires uniform attraction of an open neighbourhood of A.
- weaker, measure based notions, (that allow one to ignore exceptional sets of initial conditions) are probably closer to what is wanted in many applications. (Measure attractor)
- measure attractor are not required to attract nice sets or neighbourhoods uniformly; just points from a set of positive Lebesgue measure
- positive probability of observing the system evolve in this way E.g. in numerical experiments
- other notions are possible E.g. statistical attractors

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Measure and statistical attractors

The basin of attraction of a compact invariant set A is

$$\mathcal{B}(A) = \{x \ : \ \lim_{t\to\infty} d(\phi(t,x), A) = 0\}.$$

The basin of statistical attraction of a compact invariant set A is

$$\mathcal{B}_{\text{stat}}(A) = \{x \ : \ \lim_{s\to\infty} \frac{1}{s}\ell\{0 \le t \le s; \phi(t,x) \in B_\epsilon(A)\} = 1 \text{ for all } \epsilon > 0\}.$$

- A is said to be a *measure attractor* if $\ell(\mathcal{B}(A)) > 0$.
- A is said to be a *statistical attractor* if $\ell(\mathcal{B}_{\text{stat}}(A)) > 0$.

A statistical attractor can be characterised as attracting almost all the time:

- We say that a measurable set $M \subset \mathbb{R}$ has full density at ∞ if

  $$\lim_{s\to\infty} \frac{1}{s}\ell(M \cap [0,s]) = 1$$

  .
- Then for every $x \in \mathcal{B}_{stat}(A)$, there exists a set T_∞ of full density at ∞ such that

  $$\lim_{t\to\infty,\, t\in T_\infty} d(\phi_t(x), A) = 0\,.$$

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Measure attractor example I: homoclinic and heteroclinic cycles

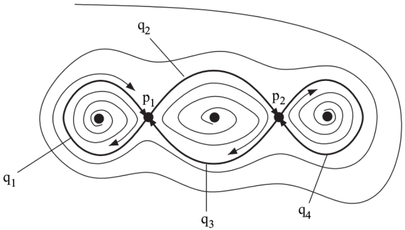

Figure: sourced from "On statistical attractors and the convergence of time averages", Ö. Karabacak, P. Ashwin .Math. Proc. Camb. Phil. Soc. (2011)

- The set composed of the union of $\{p_1, p_2, q_1, q_2, q_3, q_4\}$ is invariant and an attractor.
- Invariant sets satisfying the measure attractor definition are $\{p_1, q_1\}$, $\{p_2, q_4\}$, $\{p_1, q_2, p_2, q_3\}$ and other combinations
- Statistical attractors: $\{p_1, p_2\}$, $\{p_1\}$ and $\{p_2\}$ etc

Measure attractors can be stranger still: riddled basins

System of ODEs composed of a scalar system with a subcritical pitchfork coupled to the Lorenz '84 model.

$$\dot{x} = -y^2 - z^2 - ax + aF$$
$$\dot{y} = xy - bxz - y + G$$
$$\dot{z} = bxy + xz - z$$
$$\dot{w} = (x - \lambda)w + w^3 - cw^5,$$

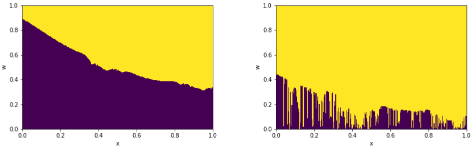

Figure: $a = 0.25$, $b = 4$, $F = 8$, $G = 1$, $c = 0.1$. We note that $w = 0$ is an invariant subspace for all parameter values, but whether it is attracting or repelling depends on the parameter λ.. If the solution is found to converge to $w = 0$ (that is, the w coordinate is measured at less than 0.01 distance away from 0), the initial condition is coloured purple, otherwise it's coloured yellow. LHS: $\lambda = 1.20$ illustrates an asymptotically stable basin of attraction for $w = 0$. RHS: $\lambda = 1.05$ shows a riddled basin of attraction for the $w = 0$ attractor

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Nonautonomous attractors

A *nonautonomous* set $\mathcal{A}$ is a family of sets $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$. It is said to be invariant for process ψ if $\psi(t, s, A(s)) = A(t)$.

Definition

We say $\mathcal{A}$ is a local pullback attractor *if there is an $\eta > 0$ such that*

$$\lim_{t_0 \to -\infty} d(\psi_{t,t_0} B_\eta(A(t_0)), A(t)) = 0$$

for all $t \leq 0$.

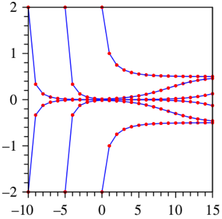
Correspondingly, we say $\mathcal{A}$ a local forward attractor *if there is an $\eta > 0$ such that*

$$\lim_{t \to \infty} d(\psi_{t,t_0} B_\eta(A(t_0)), A(t)) = 0$$

for all $t_0 \geq 0$.

Both types of convergence are important

- Both forward and pullback notion of attractor involve attraction of a uniform neighbourhood (of a nonautonomous set)
- Forward and pullback notions are not necessarily related: there are examples of a local pullback attractor that repels in future time. E.g.



$$x_{k+1} = \frac{\lambda_k x_k}{1+|x_k|}, \quad \lambda_k := \begin{cases} \lambda, & k \geq 0, \\ \lambda^{-1}, & k < 0 \end{cases}$$

Figure: sourced from "Limitations of pullback attractors for processes", Peter E. Kloeden , Christian Pötzsche, Martin Rasmussen, Journal of Difference Equations and Applications

- Future behaviour may depend on the present state; pullback attractors tell us likely present state of a system that has started in some distant past.
- Both types of convergence are important to understand the dynamics.

Nonautonomous measure attractors

- Difficulty in defining a pullback nonautonomous attractor in a measure theoretic way is due to the difficulty in defining a suitable ‘basin of attraction’.
- In a pullback sense, the set of points that converge to a nonautonomous set, ‘live’ in the infinite past.
- Furthermore, such a set is likely to be a nonautonomous set itself. In fact, it it possible that a system has no ‘deterministic’ basin, as the following example shows.

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Example: Nonautonomous system with no deterministic basin

Consider the process given by

$$\Phi_{t,t_0}(x) := \frac{(x + a\sin t_0)}{\sqrt{1 + ((x + a\sin t_0)^{-2} - 1)e^{2(t-t_0)}}} - a\sin t.$$

- for convergence we require $-1 - a\sin t_0 < x < 1 - a\sin t_0$.
- if $a < 1$, then the set $(-1 + a, 1 - a)$ is a deterministic pullback basin of attraction.
- $D(t_0) = \{x; |x + a\sin t_0| < 1\}$ is pullback attracted to $A(t) = -a\sin t$ for all $a > 0$.

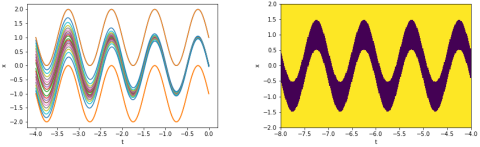

Figure: LHS: Plotted trajectories with $a = 1$, starting at $t_0 = -8\pi$ and integrated forward to time 0, with initial conditions in the range $[-1, 1]$. All trajectories, apart from those with initial conditions $x_0 = \pm 1$ converge to $-a\sin t$. However, starting at different initial times would change the range of initial conditions that converge. This is shown in the second plot (RHS). Initial conditions at different starting times $t_0 \in [-16\pi, -8\pi]$ are integrated forward to $t_0 + 8\pi$. Initial points that converge to $-a\sin t$ are coloured in purple, while those that diverge are in yellow.

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Forward measure attractor definition

If $\mathcal{A}$ is a compact, invariant nonautonomous set, one can conveniently define a *basin of forward attraction* to be the nonautonomous set $\mathcal{B}^+(\mathcal{A})$ with fibres

$$\mathcal{B}^+(\mathcal{A})(t_0) = \{x \ : \ \lim_{t\to\infty} d(\Phi_{t,t_0}(x), A(t)) = 0\},$$

for all $t_0 \geq 0$.
We say a $\mathcal{A}$ is a *forward measure attractor* if there exists a $t_0 \geq 0$ such that

$$\ell(\mathcal{B}^+(\mathcal{A}))(t)) > 0$$

for all $t \geq t_0$.

Pullback measure attractor definition

We say that an invariant compact nonautonomous set $\mathcal{A}$ is a *pullback measure attractor* if there is a nonautonomous set $\mathcal{N} := \{N(t)\}_{t\in\mathbb{R}}$ such that,

1. $\liminf_{t\to-\infty} \ell(N(t)) > 0$, and
2. $$\lim_{t_0\to-\infty} d(\Phi_{t,t_0} N(t_0), A(t)) = 0$$

   for all $t \leq 0$.

Statistical attractor example

We identify the unit circle S^1 with the unit interval $[0,1]$ and consider the autonomous differential equation

$$\dot{\theta} = \begin{cases} y^3 & : \quad \theta \leq y \\ 1 & : \quad \theta > y \end{cases} \qquad (1)$$
$$\dot{y} = -y^2,$$

which is piecewise smooth and defined on $S^1 \times \mathbb{R}^+$.

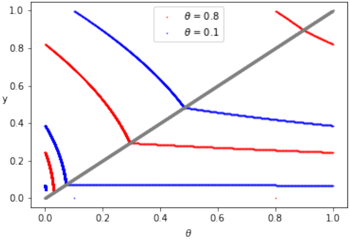

Figure: Two trajectories of (1) *for initial conditions* $(\theta, y) = (0.8, 1)$*, in red, and* $(\theta, y) = (0.1, 1)$*, in blue.*