Characterisation of structures emerging from random colouring processes on a spatial graph

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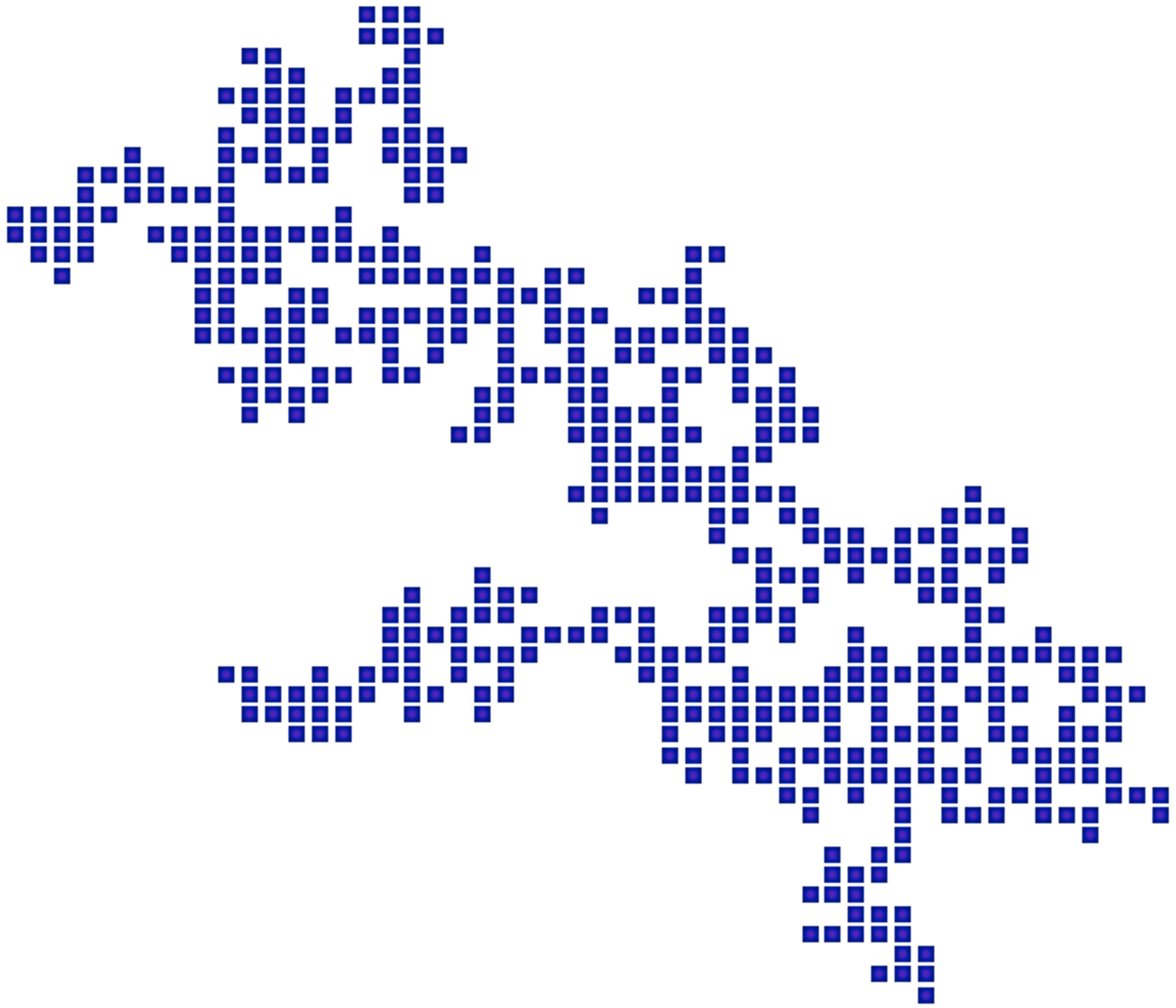

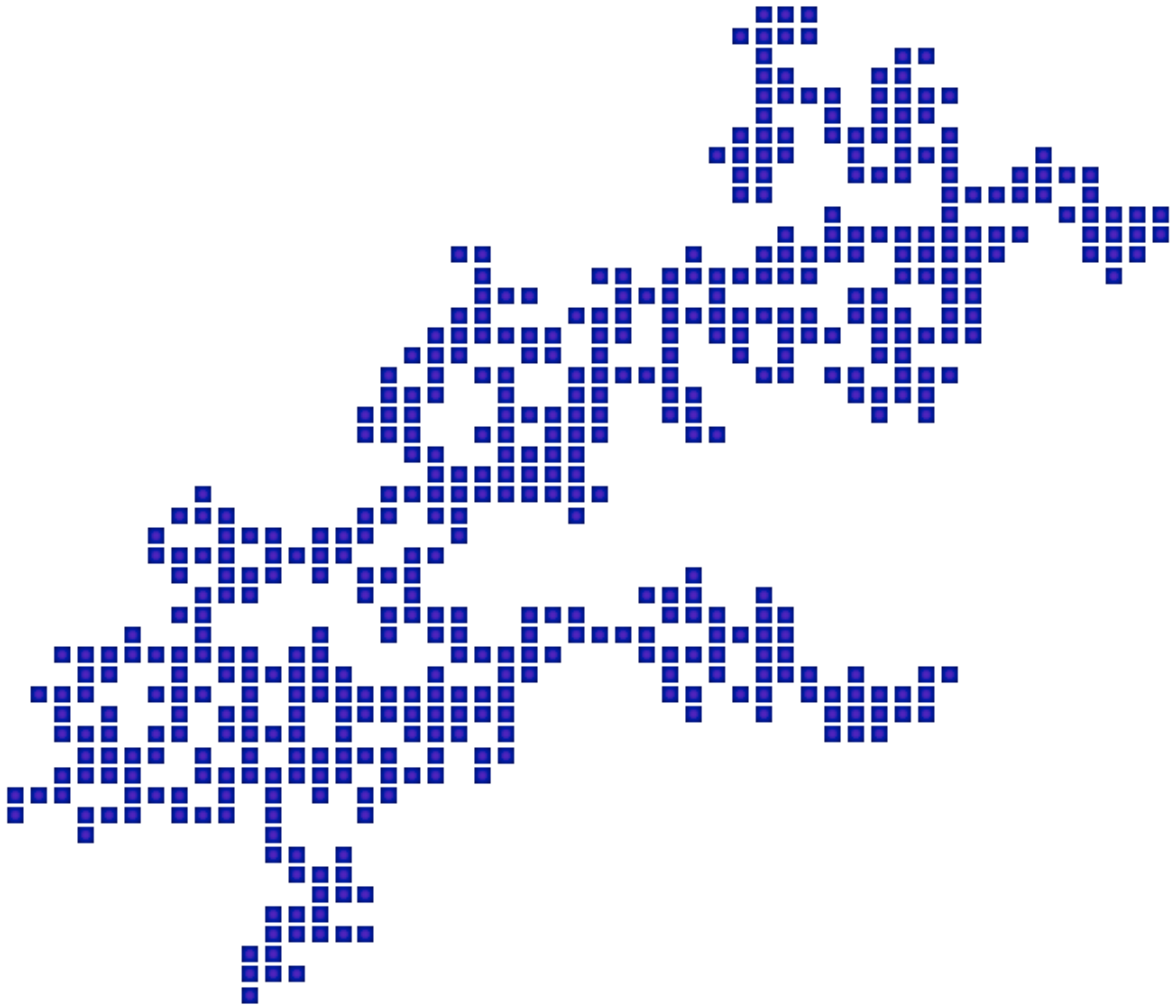

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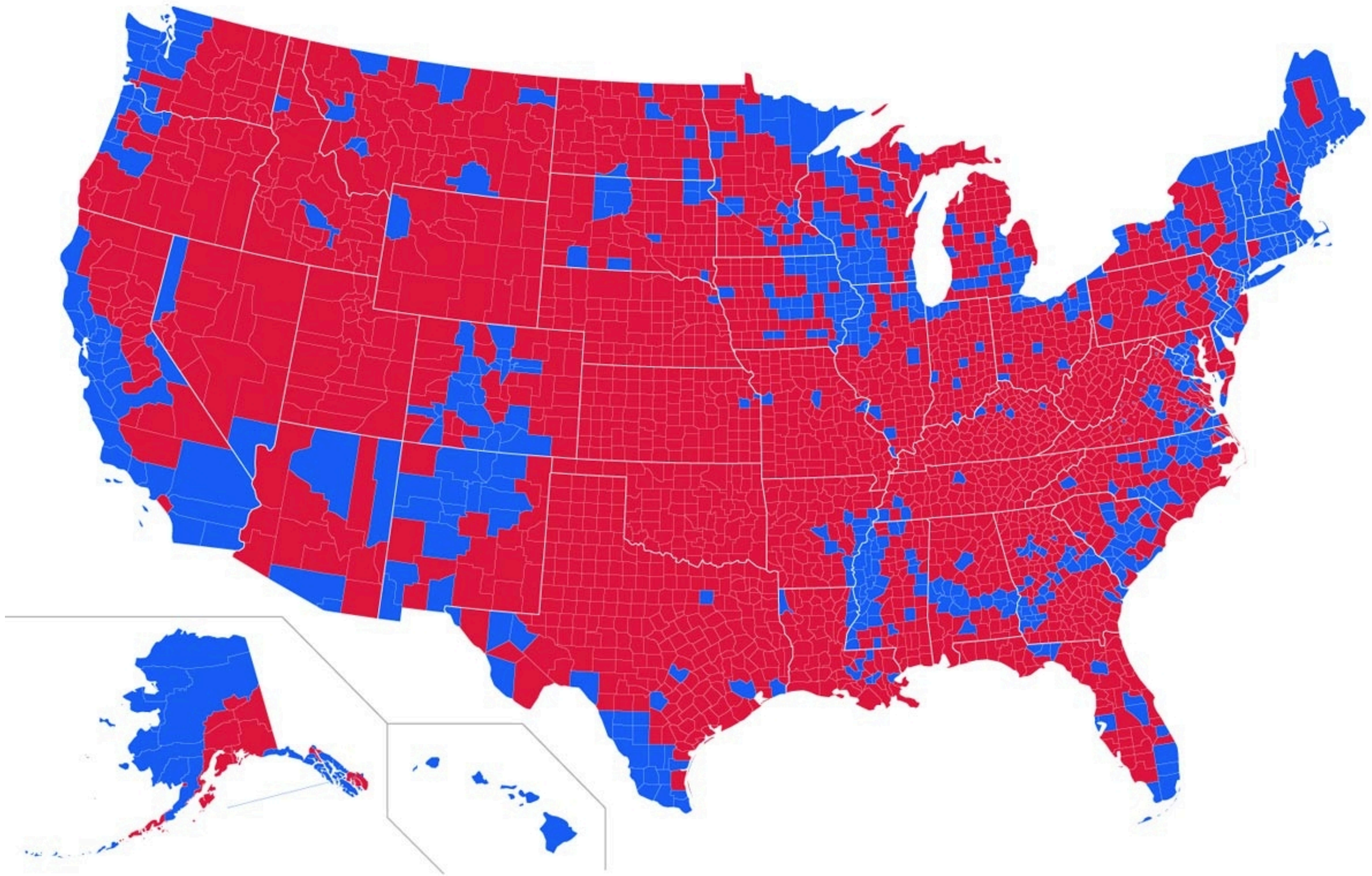
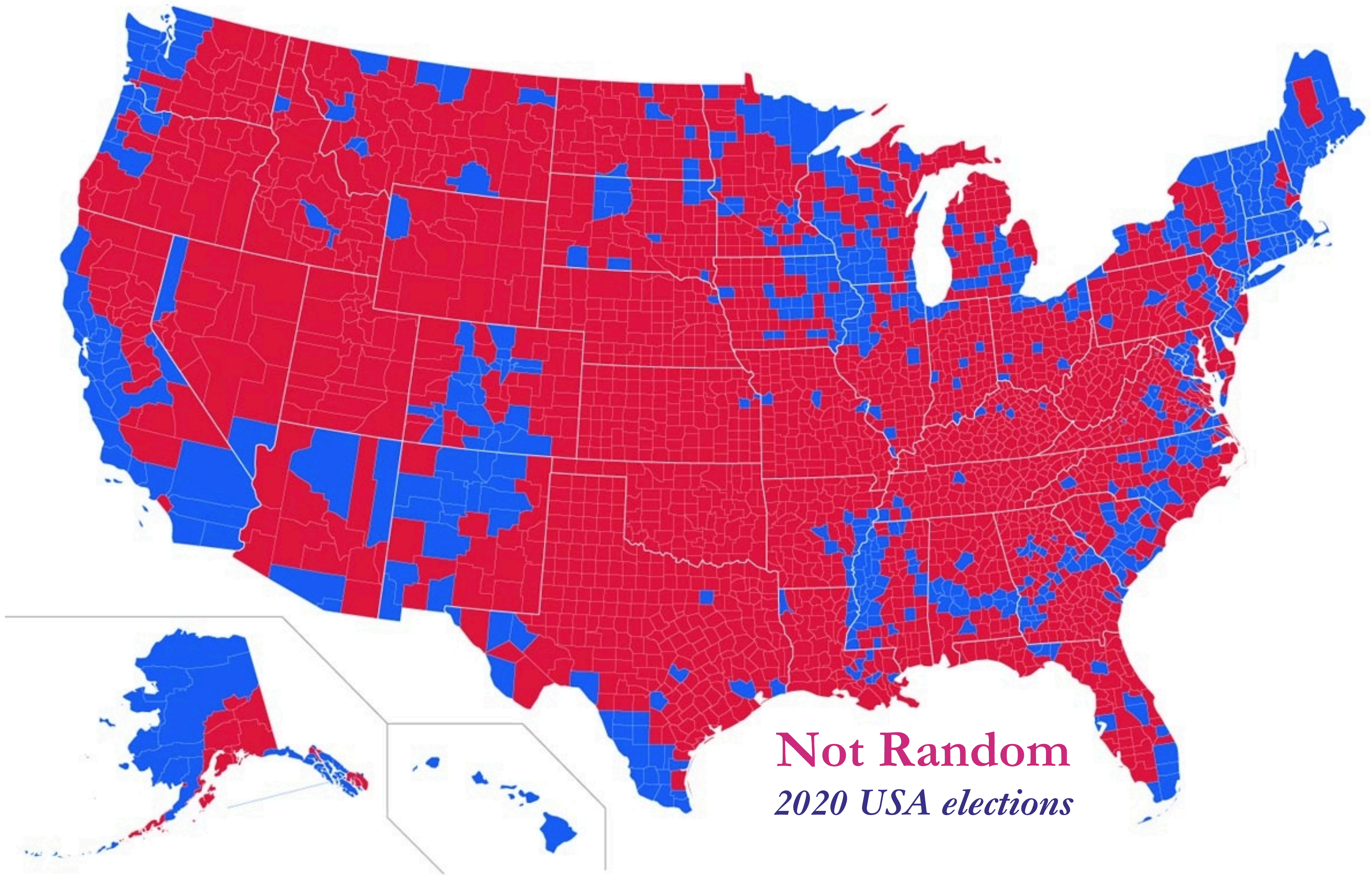
Random and not-random



Random and not-random



Not Random

2020 USA elections

Random

20x20 sites grid
randomly coloured with 2 colours

Random and not-random

- We consider a bi-dimensional squared lattice.
- We assigned to each site a colour with probability $p = \dfrac{1}{C}$
- We call this process ***random colouring process R***.
- Also in this simple case, we observe *structures*.

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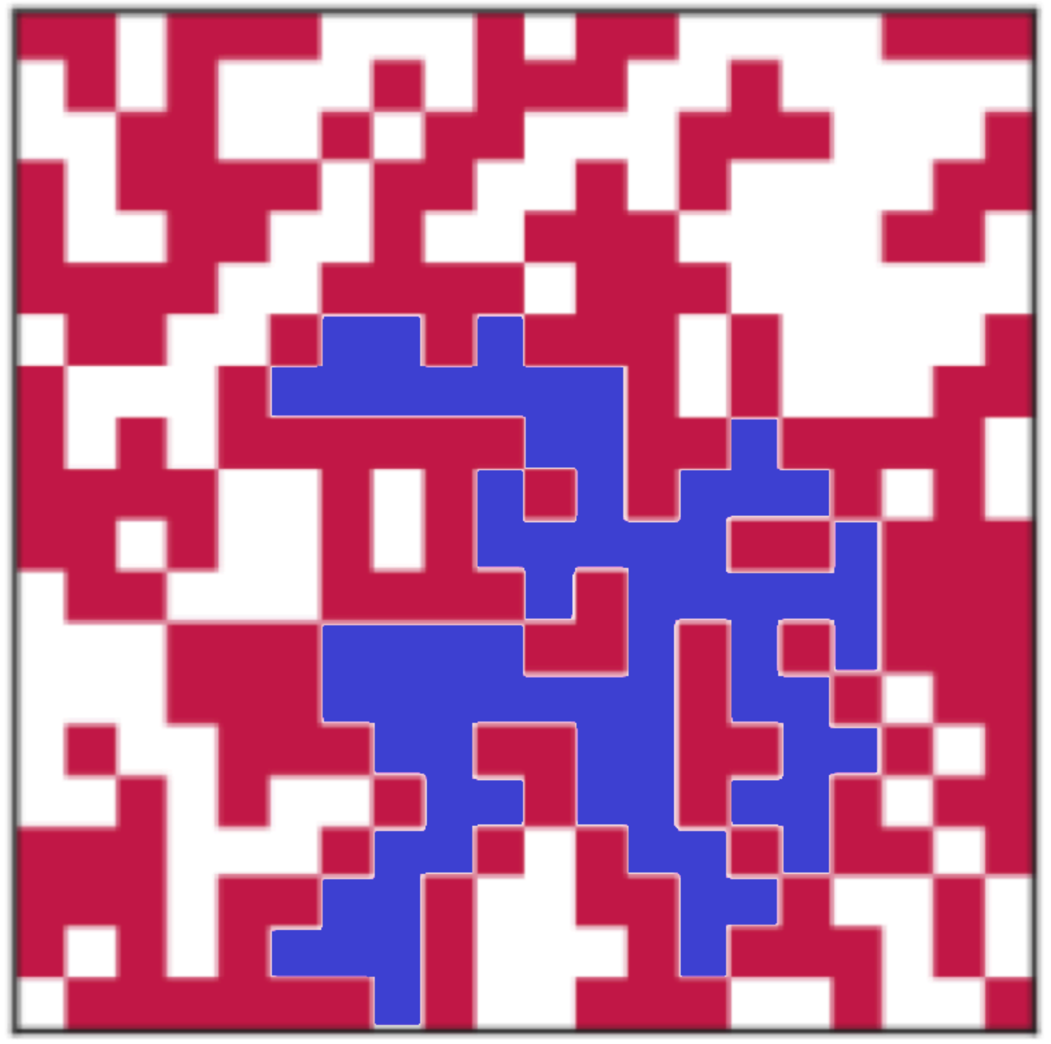
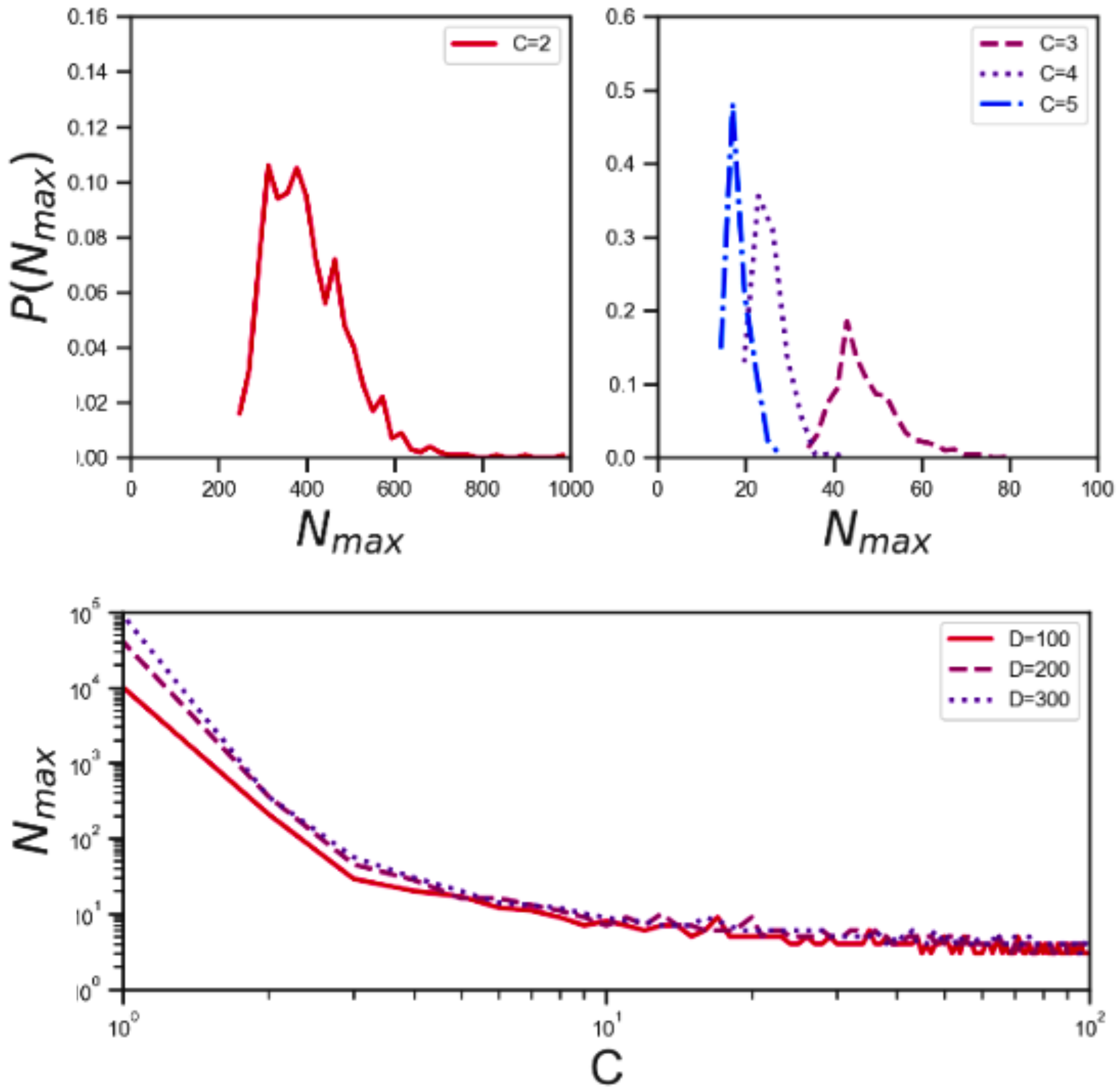


Random and not-random



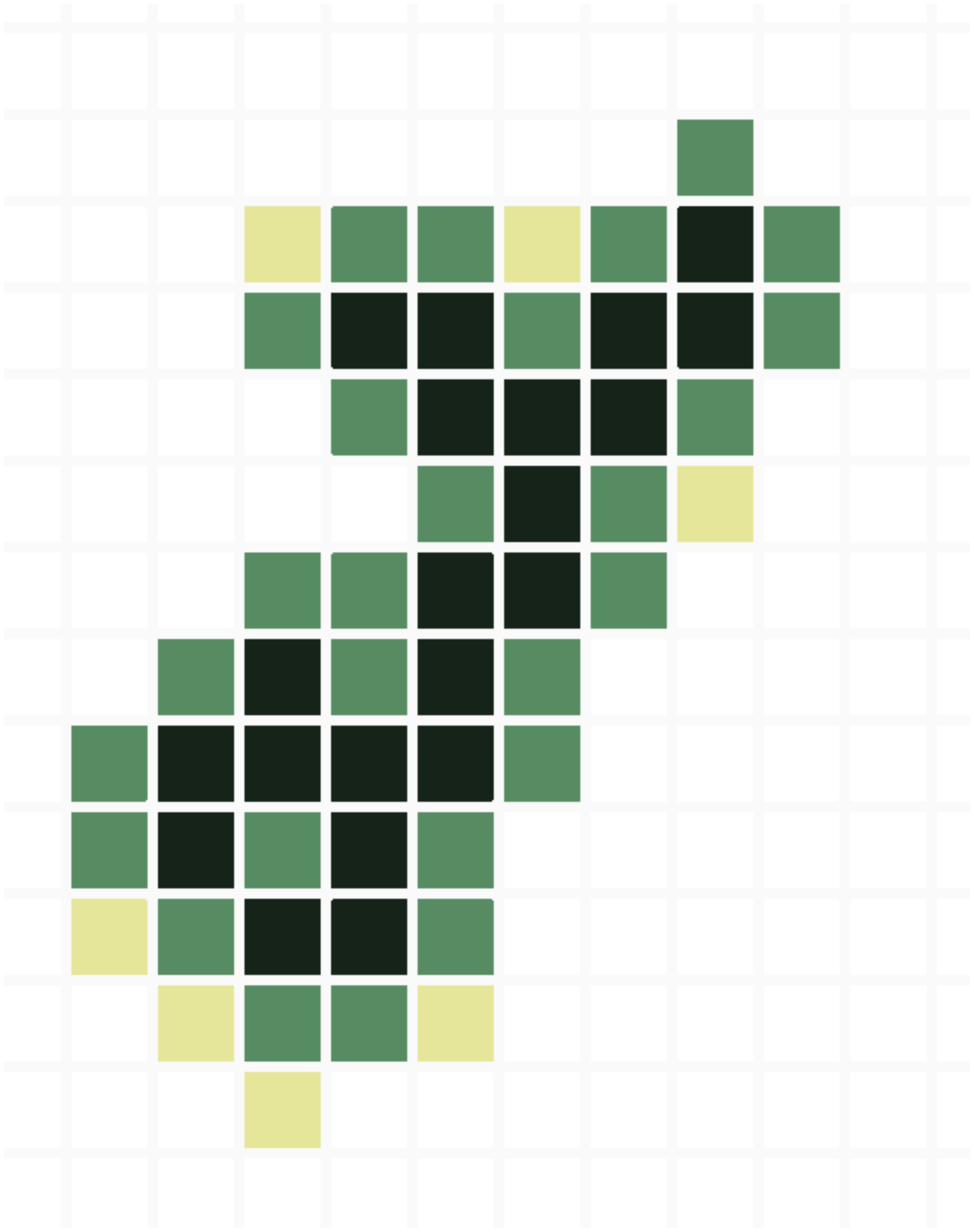
- We investigate the ***probability distribution of*** N_{max}, i.e. the dimension of the largest structure observed in a single realisation of the process R.

- When C decreases, the probability to observe large structures ***increases***.

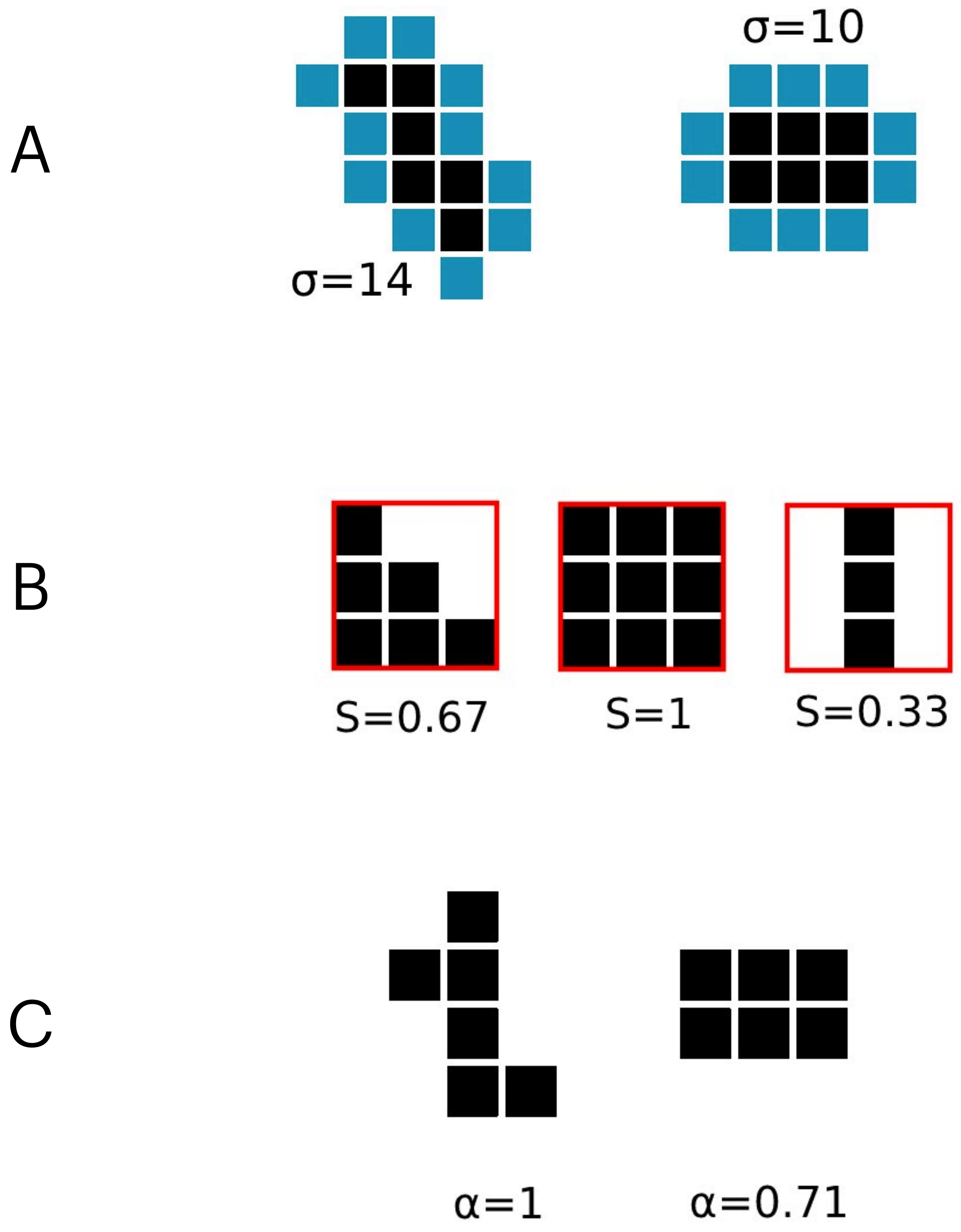
- The presence of structures is ***independent*** from the side D of the lattice.

Structural quantities

- A ***cluster*** is the maximal set of connected sites with the same colour.
- The ***size*** is the defined as the number of sites in this set and it is equal to N.
- The ***frontier*** of the cluster is the set of adjacent sites with a different colour.
- The ***size of the frontier*** is the defined as the number of sites in this set and it is equal to f.



Structural quantities



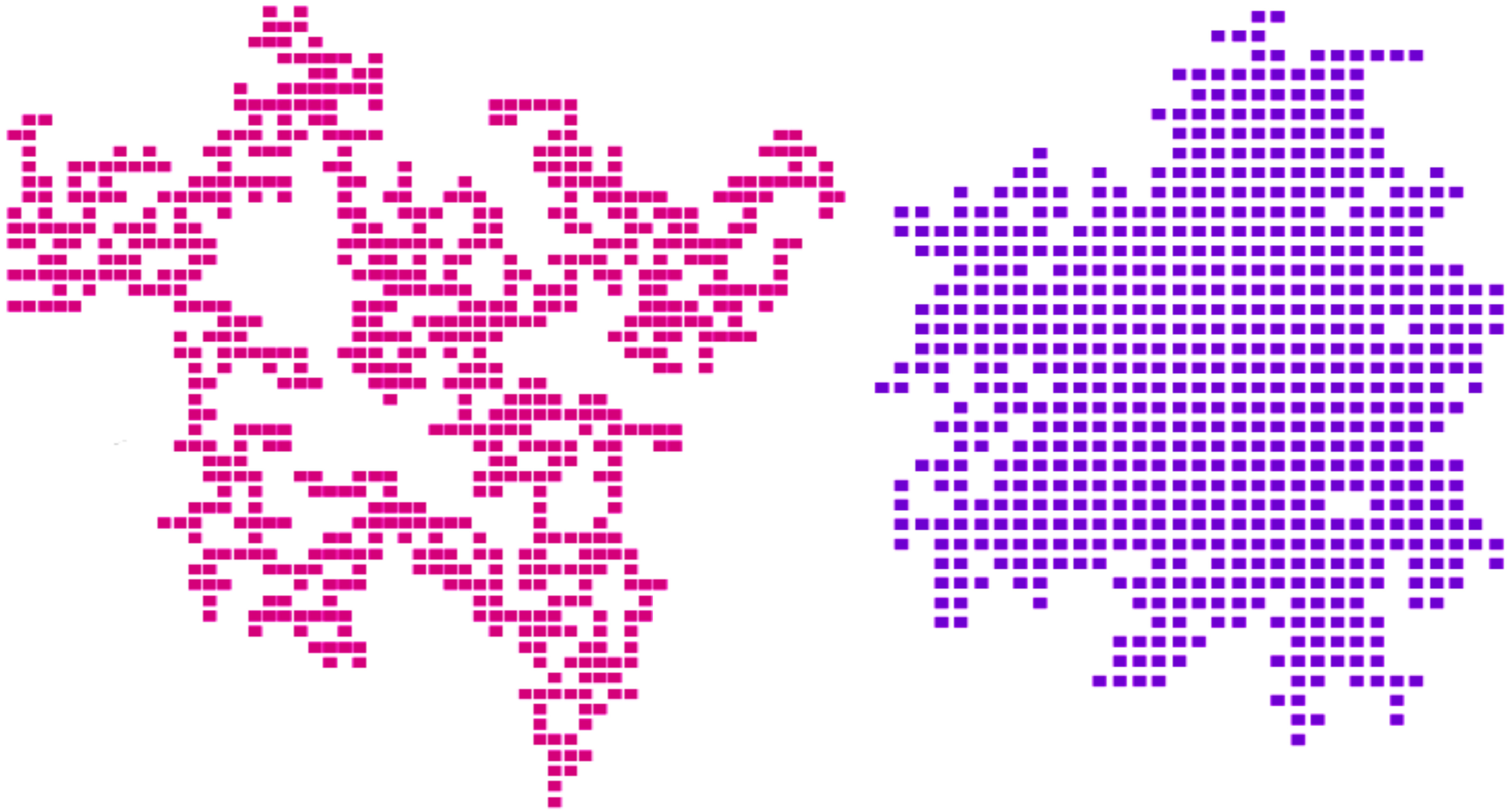
- ***Surface***: the number σ of edges shared between the cluster and the frontier.

- ***Shape factor:*** the ratio $S = \dfrac{N}{l^2}$
- $S \in [0,1]$.
- High S is related to a dense structure.

- ***Tree-likeness:*** the ratio $\alpha = \dfrac{N-1}{e_{in}}$
- $\alpha \in (0.5,1]$.
- High α is related to a large surface - small α to compact clusters.

Models

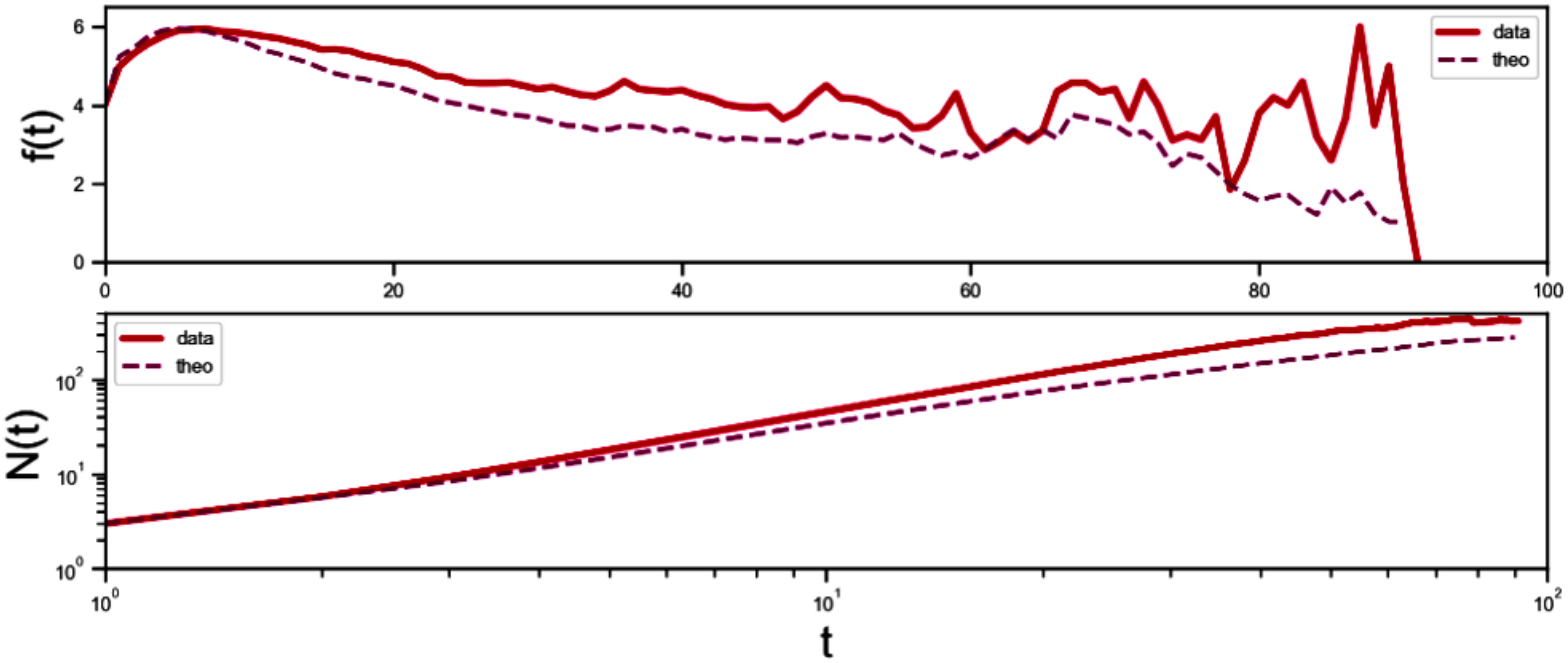
- At each time step, the frontier-sites are coloured either with the x colour, with probability $p = \dfrac{1}{C}$, or with another colour, with probability $q = 1 - p$.
- The seed cluster ***grows limited in time*** by the presence of other colours.



- At each time step, one of the frontier-sites is picked at random and coloured with the x colour with probability $p = 1$.
- The seed cluster ***grows unlimited in time***.

RGM vs EGM

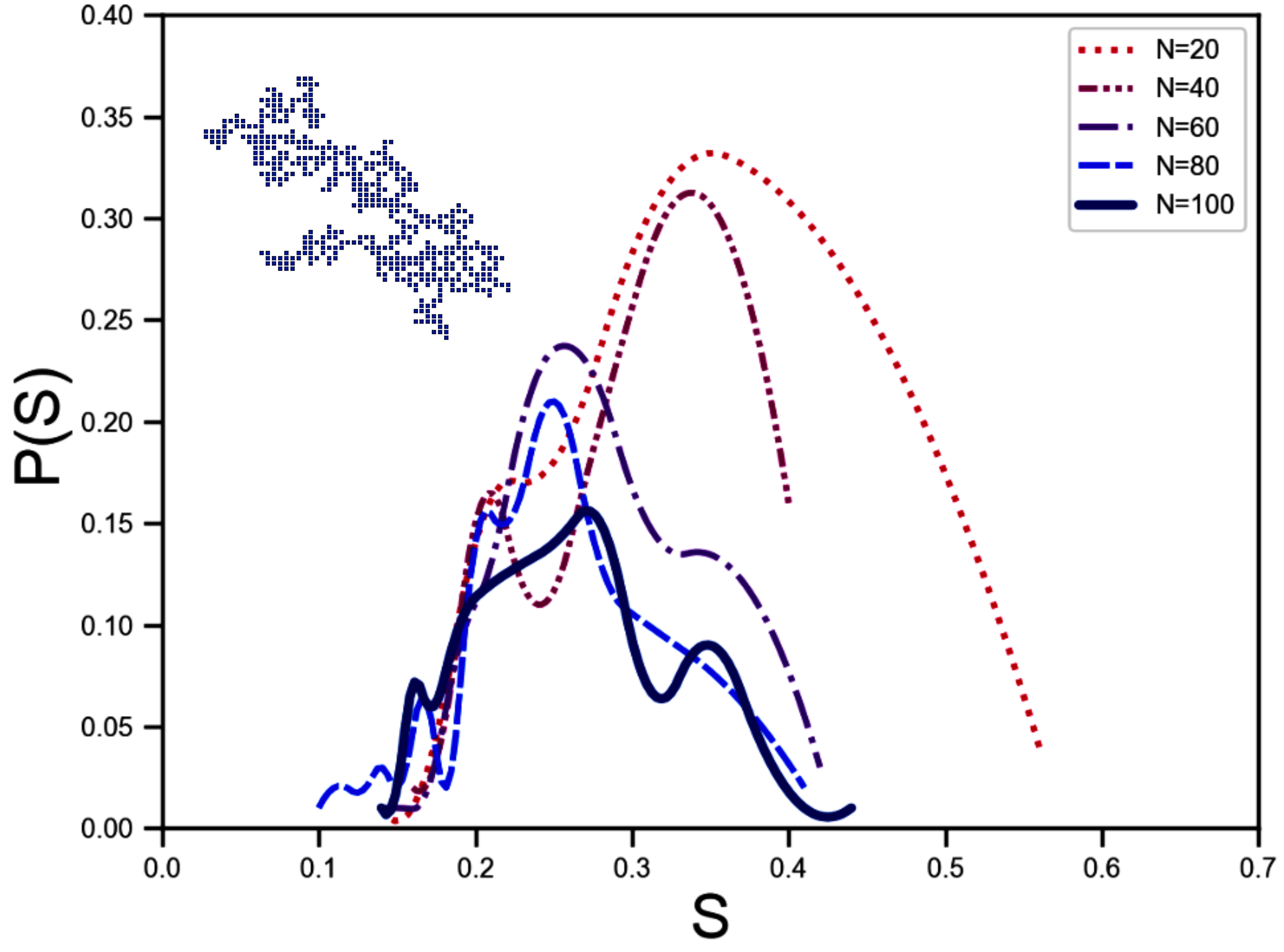
Results on structural quantities

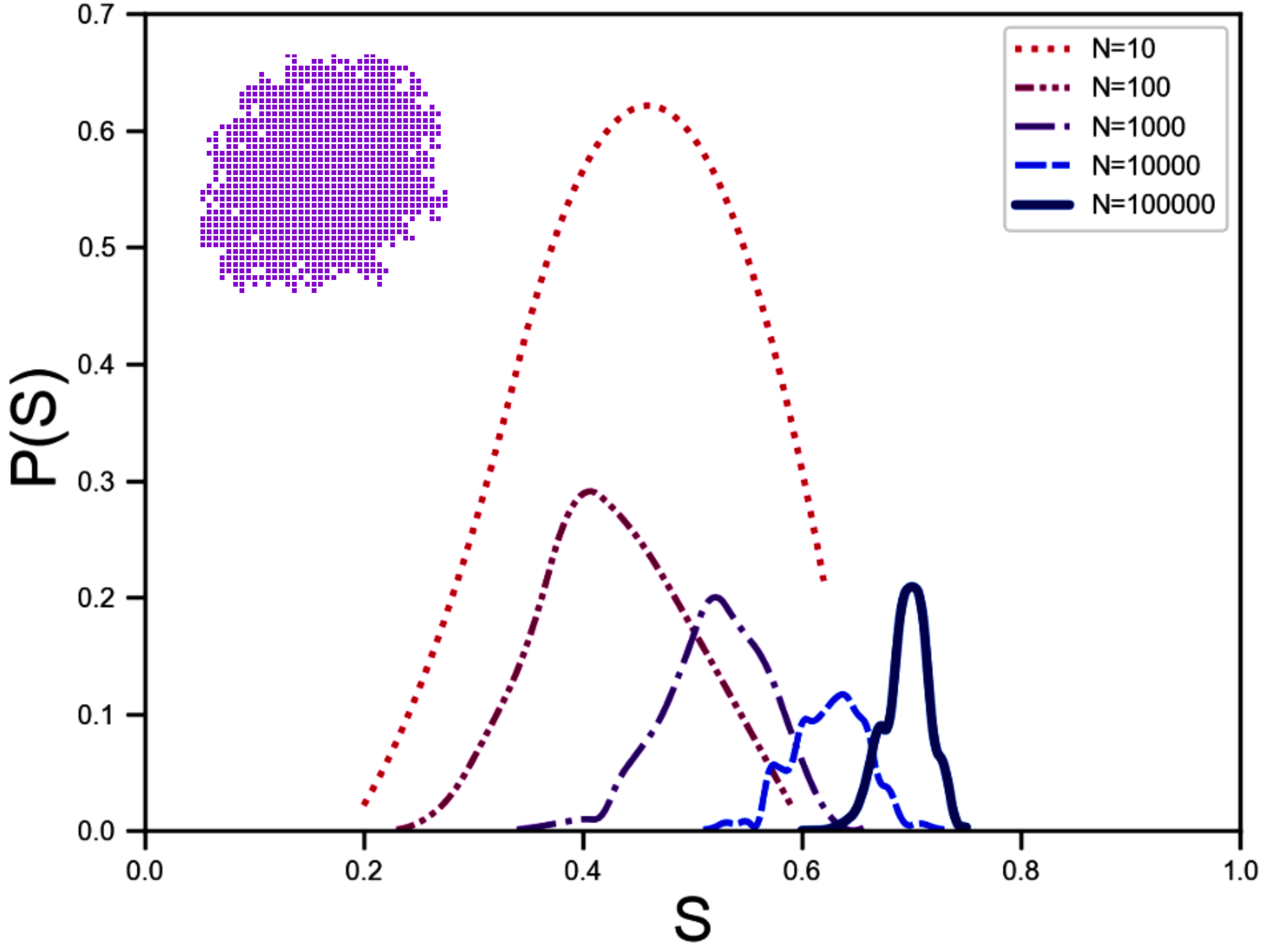


- This is the trend for the sizes N and f of RGM clusters with C=2.
- Each t represents a growth step.

- When t=[0,20], f exhibits ***a maximum***.
- As expected, ***the size N is comparable with the clusters size produced by the process R***.

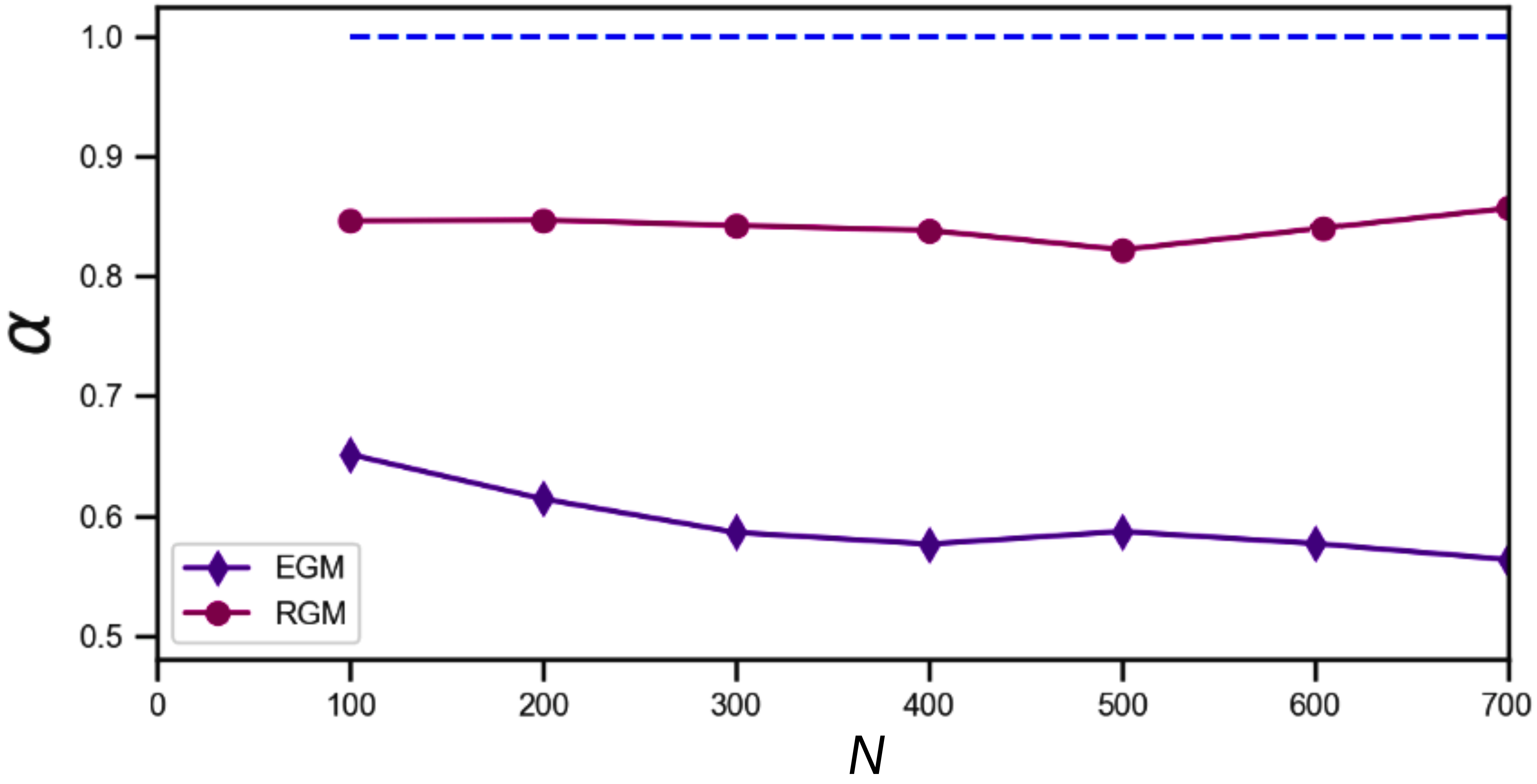
Results on structural quantities





- The peak in the distribution of shape factor for RGM clusters ***shifts to the left*** when N increases, capturing their ***rarefied structure***.
- For EGM clusters, the ***shift is to the right***, approaching the ***circle limit*** S = 0.78 for large N.

Results on structural quantities



- RGM clusters present a high tree-likeness value, because of their ***surface-like nature***.
- EGM clusters are reaching the lower limit of $\alpha = 0.5$ as expected for ***compact structures***.

Time quantities

- The exit time $\tau_{i,F}$ from a site i of the cluster $\mathscr{C}$ to one site of the frontier F, is defined as ***the number of steps a random walk needs to jump out of $\mathscr{C}$ starting from i***. In the formula, π_{ij} represents the probability to jump from the site i to the site j.

$$\tau_{i,f} = 1 + \sum_{j \in \mathscr{C}, j \neq i} \pi_{ij} \tau_{j,f}$$

- The exit time τ from a cluster $\mathscr{C}$ is the average of the exit time over all the N sites of $\mathscr{C}$. The ***mean exit time*** $\bar{\tau}$ is the average of the exit time over r realisations of $\mathscr{C}$ at fixed N.

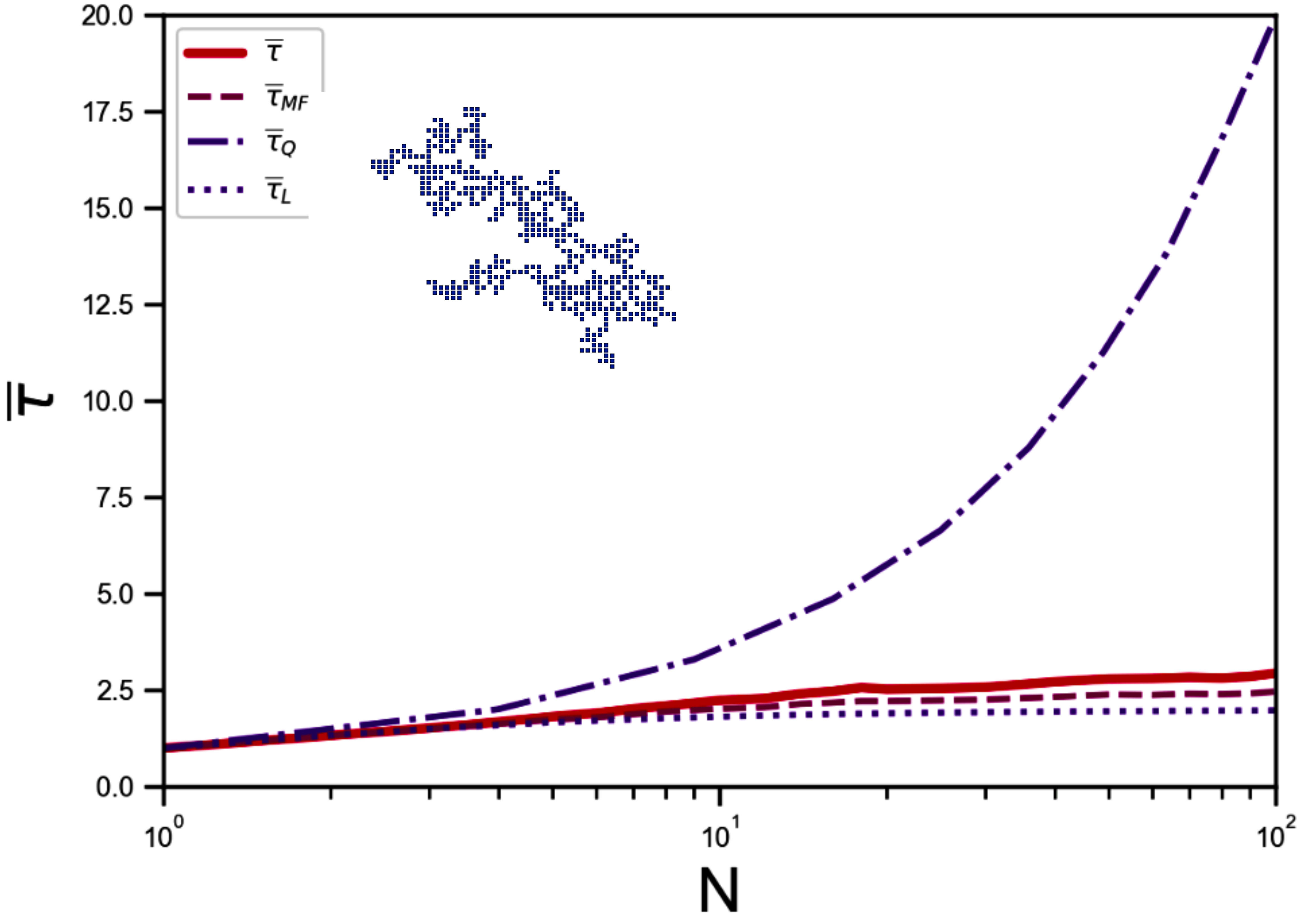
$$\tau = \frac{1}{N} \sum_{i}^{N} \tau_{i,f} \qquad \bar{\tau} = \frac{1}{r} \sum_{l}^{r} \tau_l$$

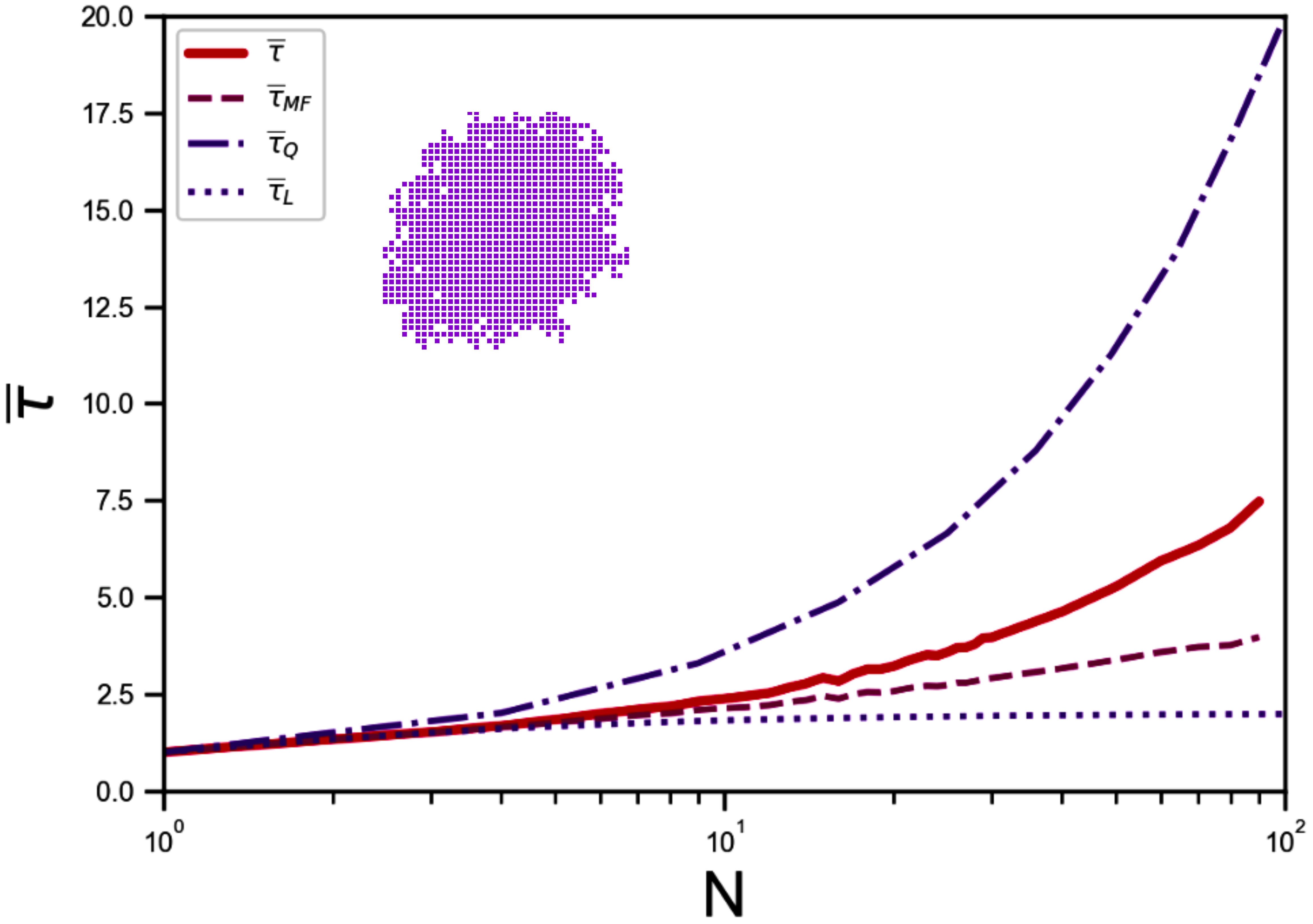
Results on time quantities

- **Intuition** : the larger the surface, the less time spent inside a cluster.
- We propose a ***Mean Field (MF) surface approximation*** for τ, as the ratio between Nk and σ, where k=4 in the case of squared lattice.

$$\tau_{MF} = \frac{Nk}{\sigma} \qquad \bar{\tau}_{MF} = \frac{1}{r}\sum_{i}^{r} \tau_{MF,i}$$

Results on time quantities

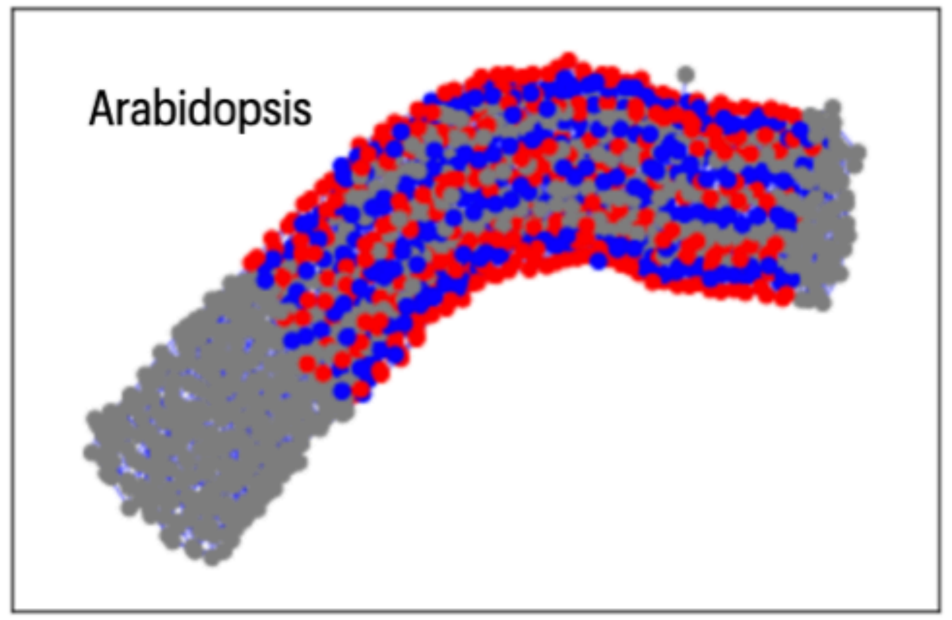


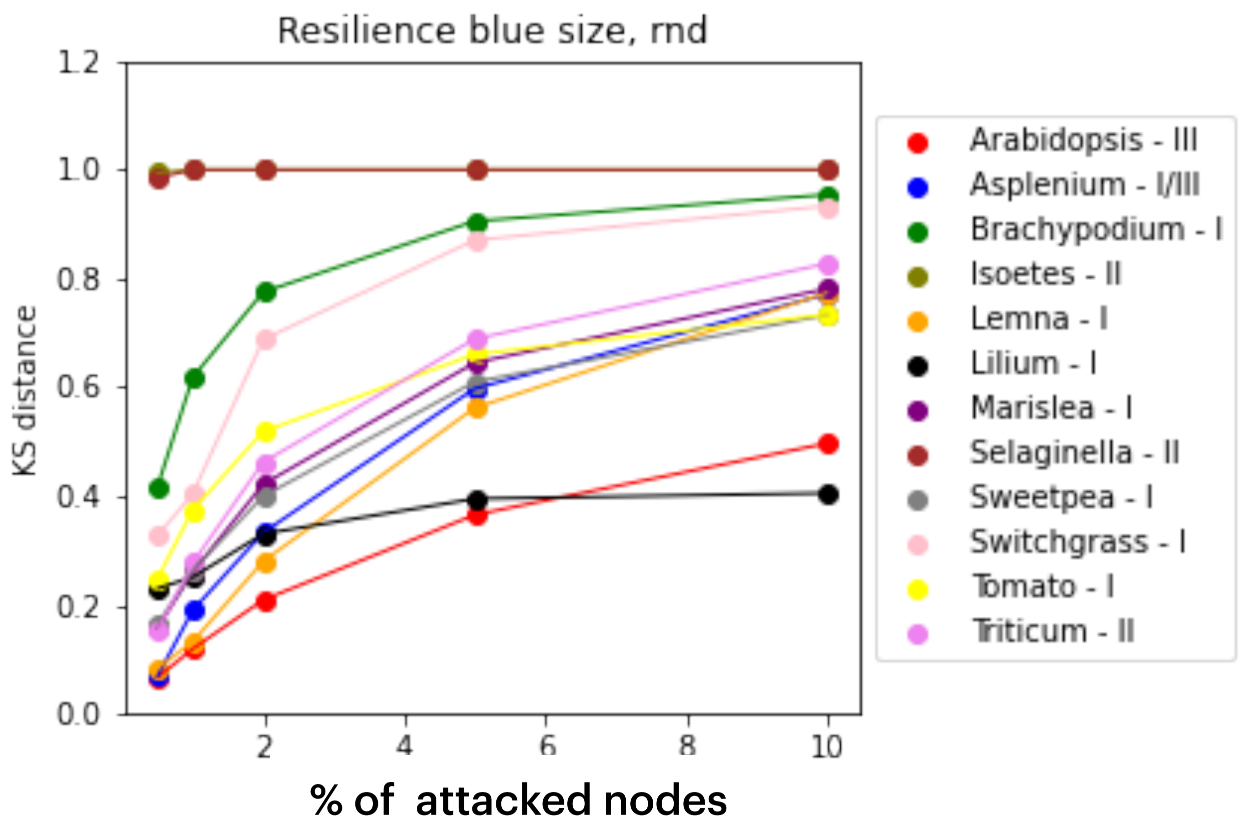


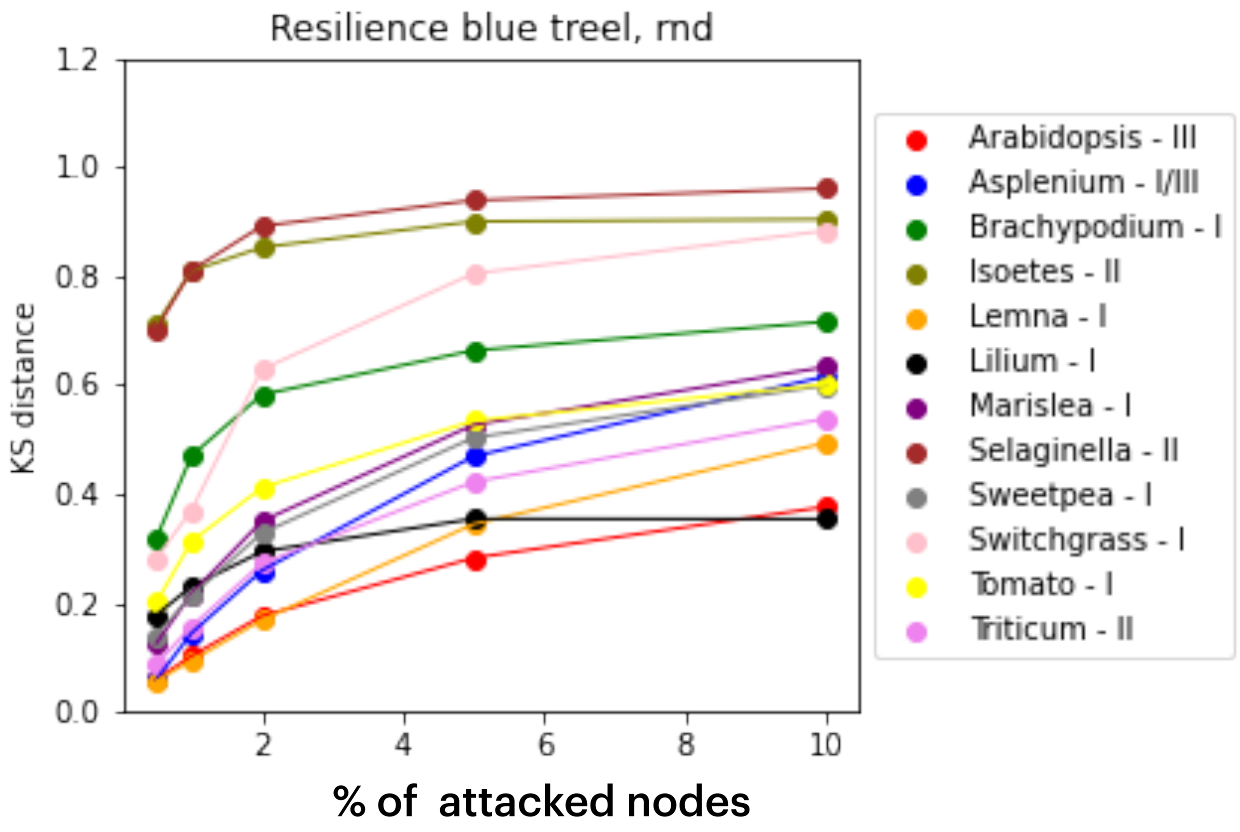
- EGM clusters exhibit a larger $\bar{\tau}$ than RGM as a consequence of their compact structure. ***The mean exit time captures the structural differences between the two families of clusters.***
- The MF surface approximation follows the data for RGM clusters but diverges for the EGM.
- $\bar{\tau}$ remains between the two theoretical limits $\bar{\tau}_Q$ and $\bar{\tau}_L$, i.e. the mean exit times from squares and lines, two families of clusters with respectively the lowest and the highest surface at fixed N.

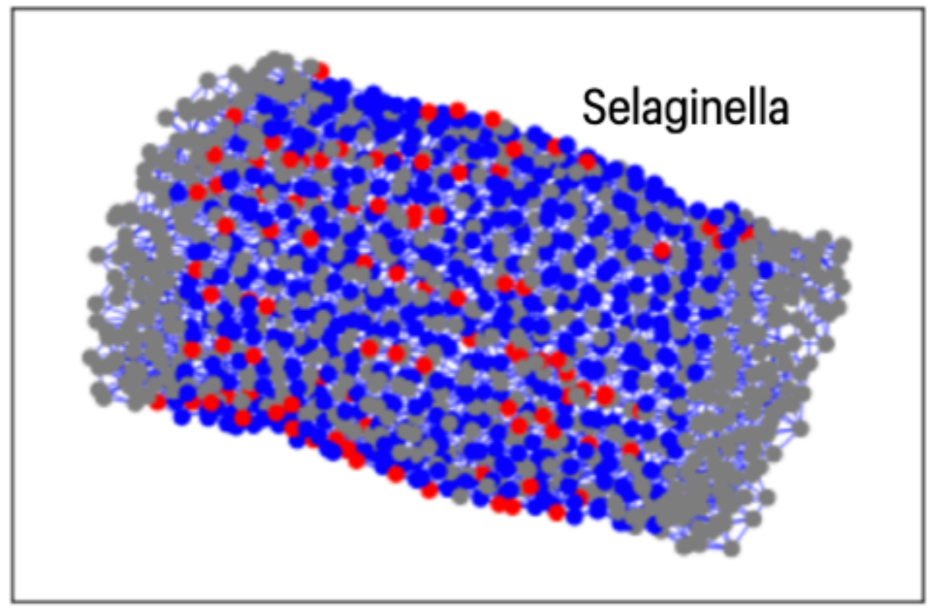
Application in real dataset

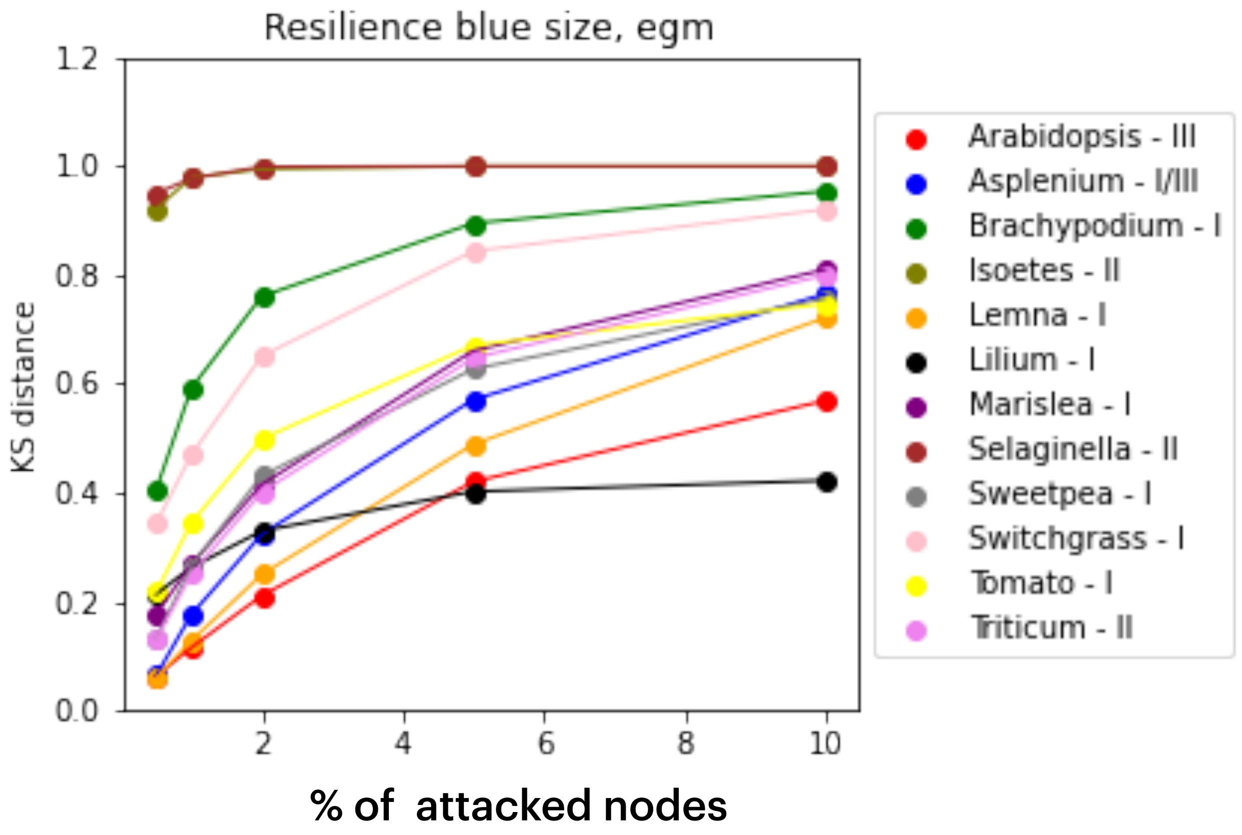
- Ongoing project on ***plant roots cells dataset***.
- The spatial network is coloured with two colours. ***Tree-likeness, size and time measures*** are collected and analysed, compared to the random case.
- Roots with more ***symmetrical spatial distribution*** are more resilient to attacks.

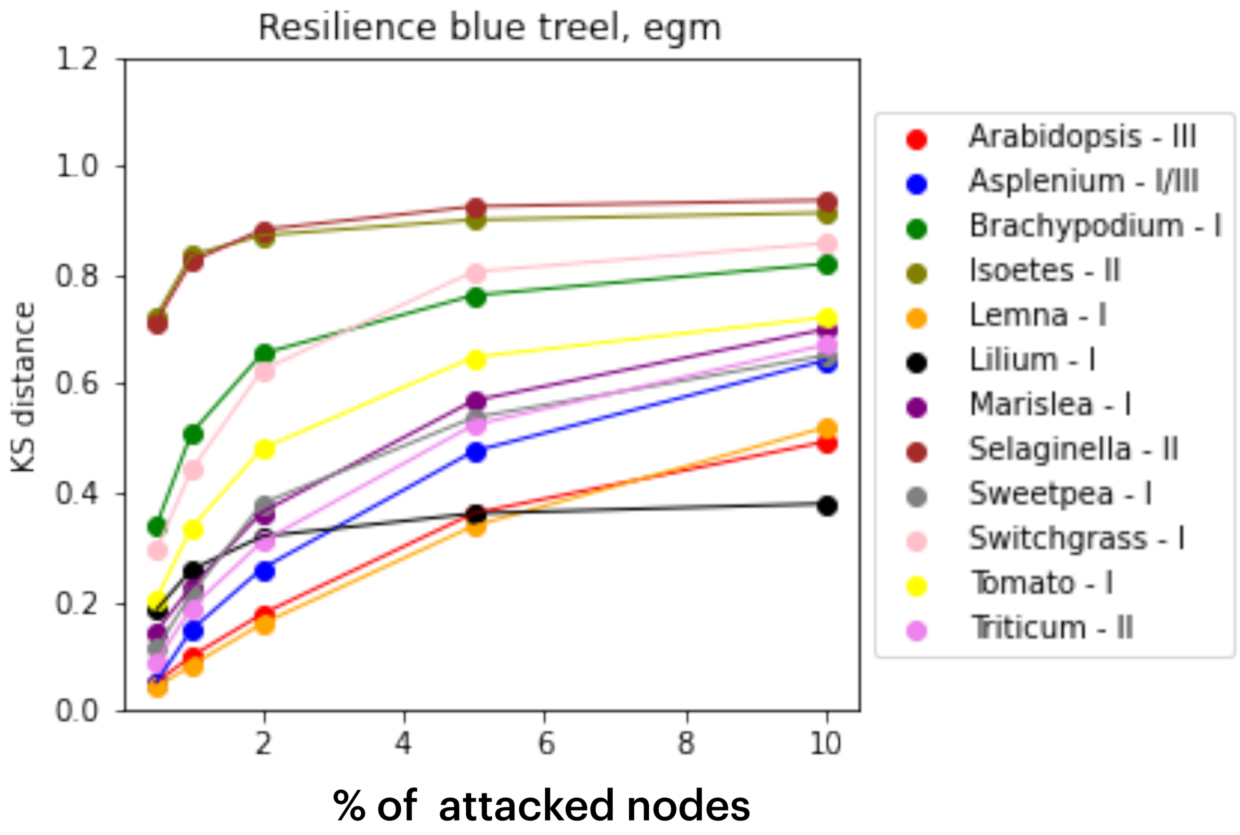






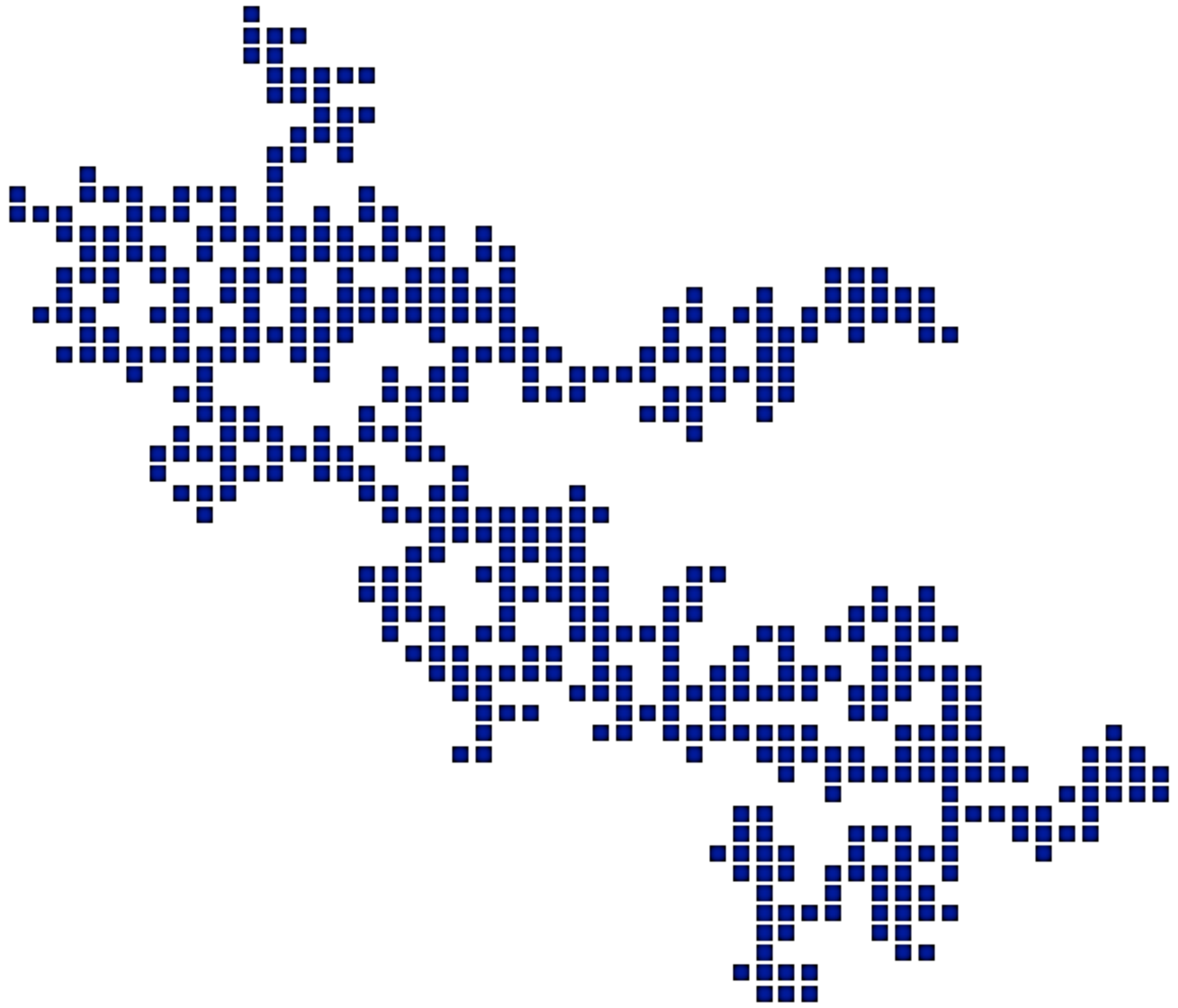






Conclusions

- The ***characterisation of random spatial structures*** is crucial to ***assign a statistical significance to the measured quantities***, when a colouring process is implemented on spatial networks.
- We show that ***the mean exit time is an effective tool to classify these clusters*** and to detect their structural properties.



Thanks for your attention!

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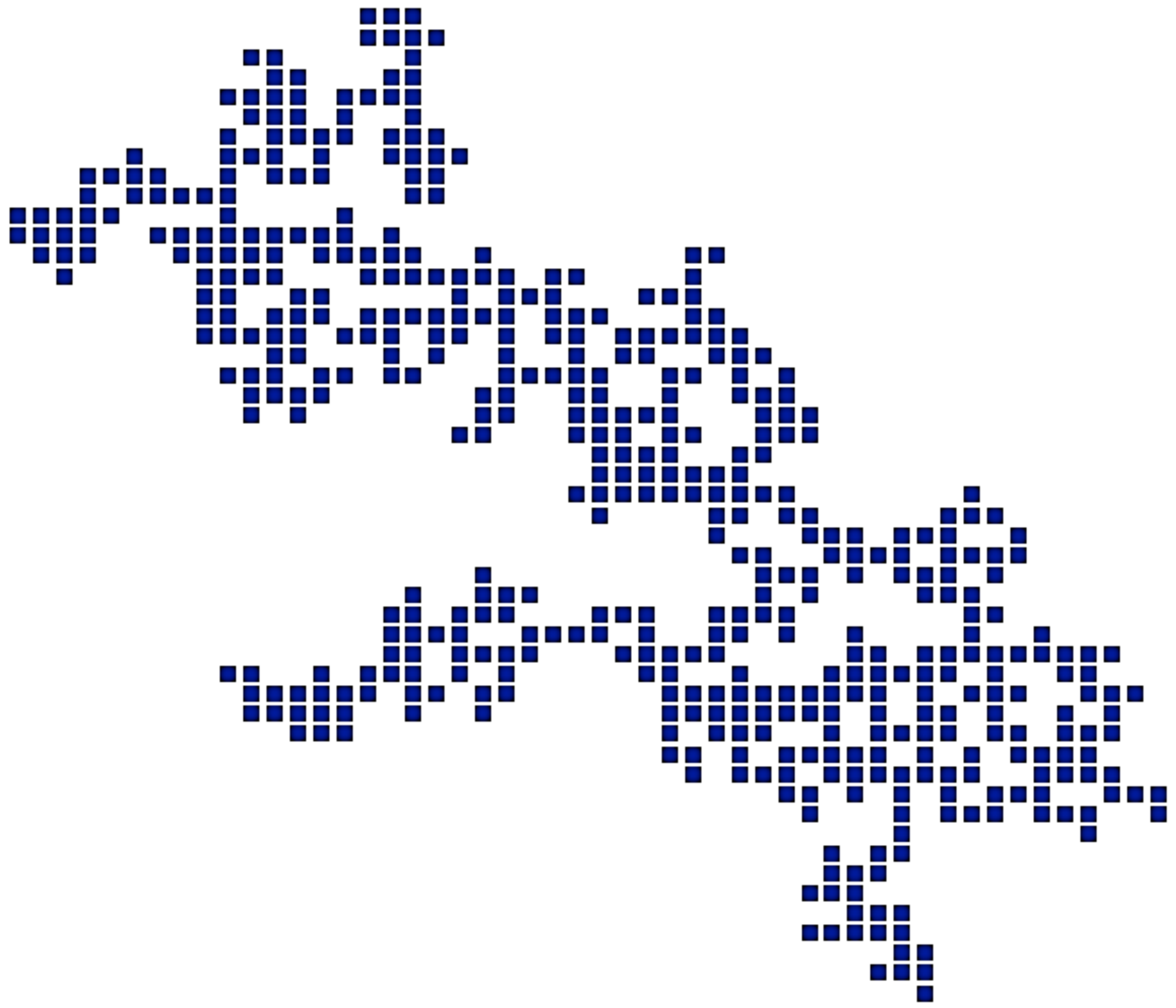

References:

[1] M. Barthelemy, Phys. Rep. 499, 1-101 (2011);
[2] M. Eden, P. Thevenaz, International Society for Optics and Photonics, 43–54 (1998);
[3] N. Masuda, M. A. Porter, R. Lambiotte, Phys. Rep. 716-717, 1–58 (2017).