Playing with Fire:

The Necessary Evil of Self-organized Criticality

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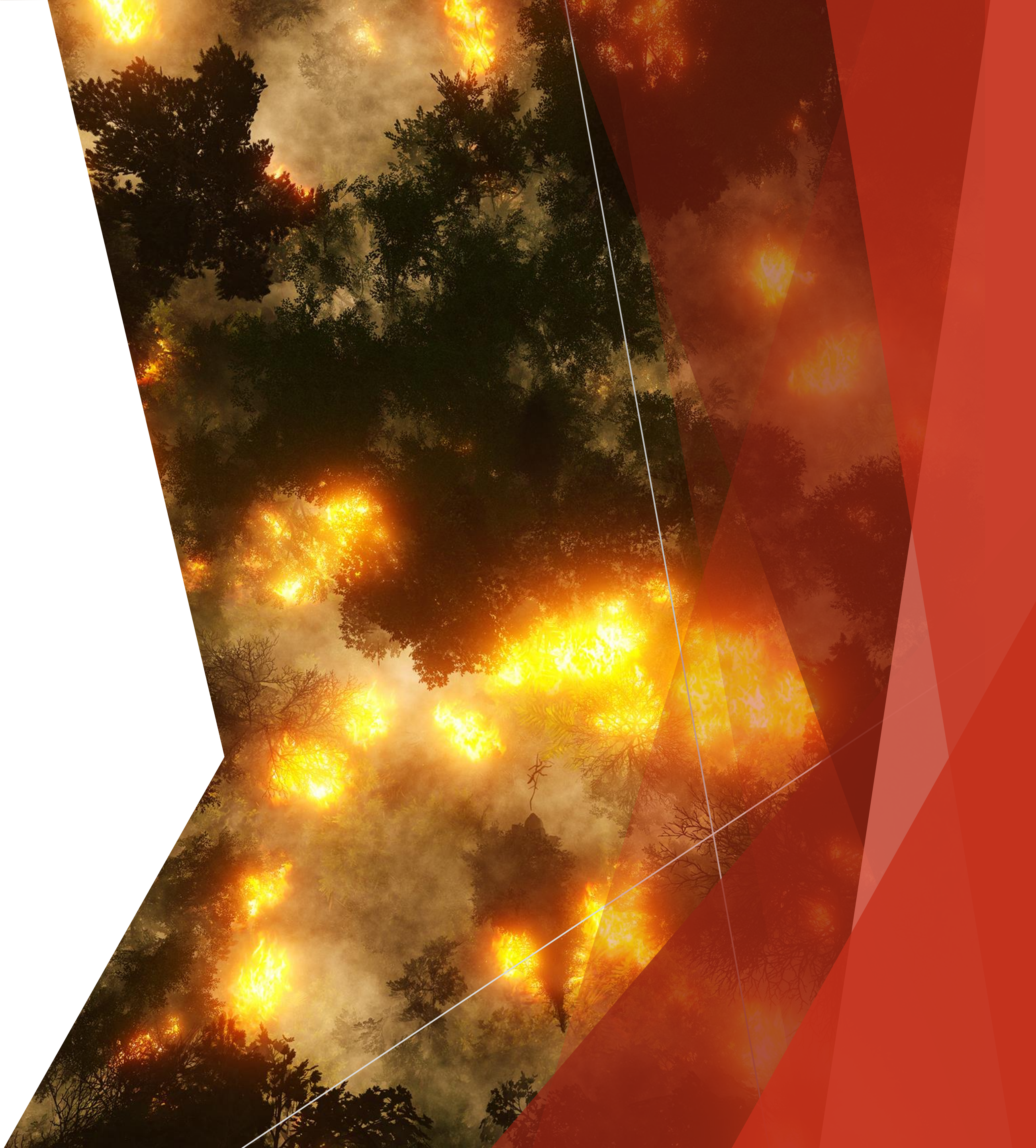
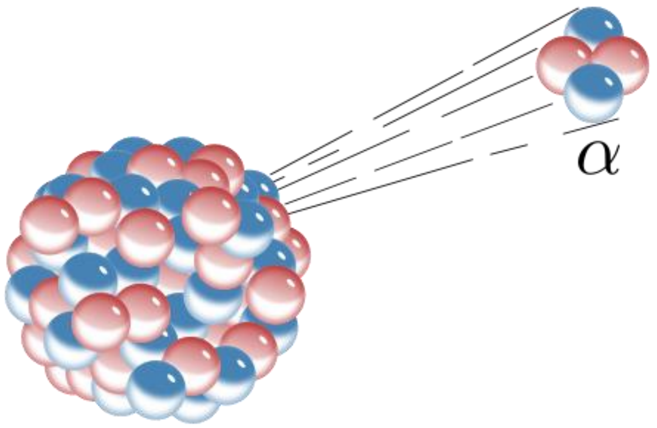
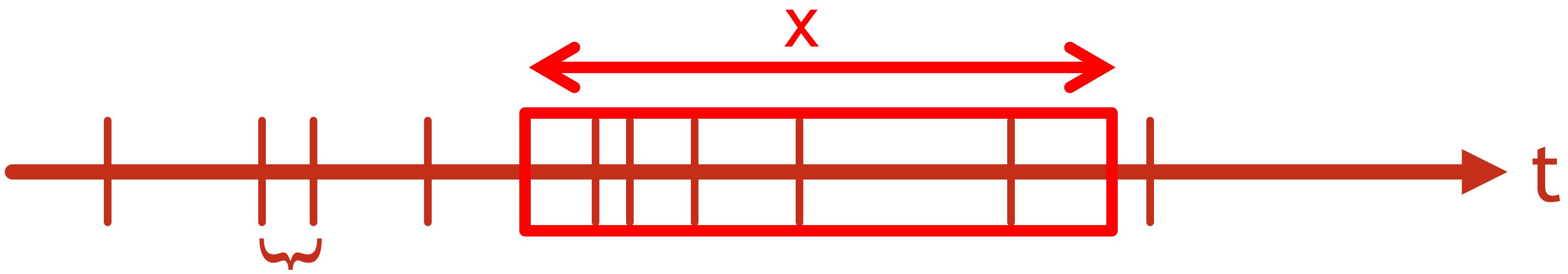


Definition

What is a Poisson Point Process?





$\sim iid\ Exponential(\lambda)$, where $\lambda > 0$

points in $[t, t + x] \sim Poisson(\lambda x)$

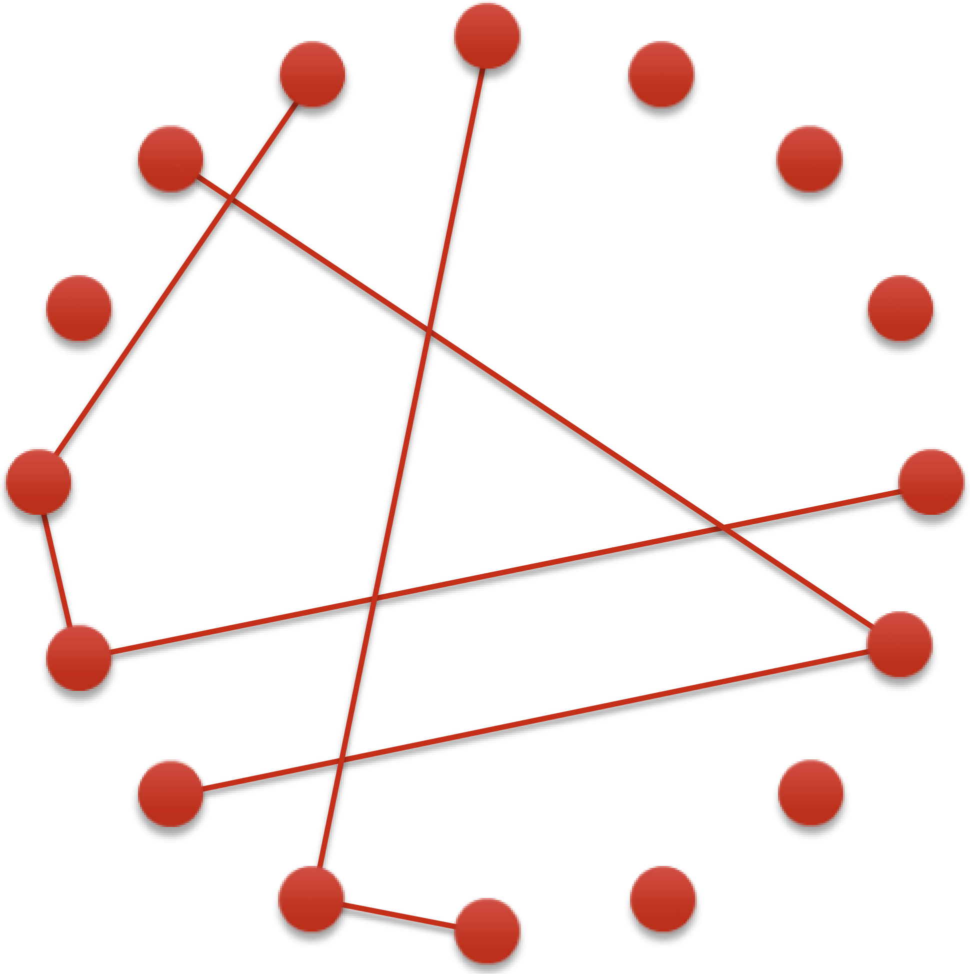
We abbreviate this to $PPP(\lambda)$.

The Mean Field Forest Fire Model

The Model

Edge Arrivals

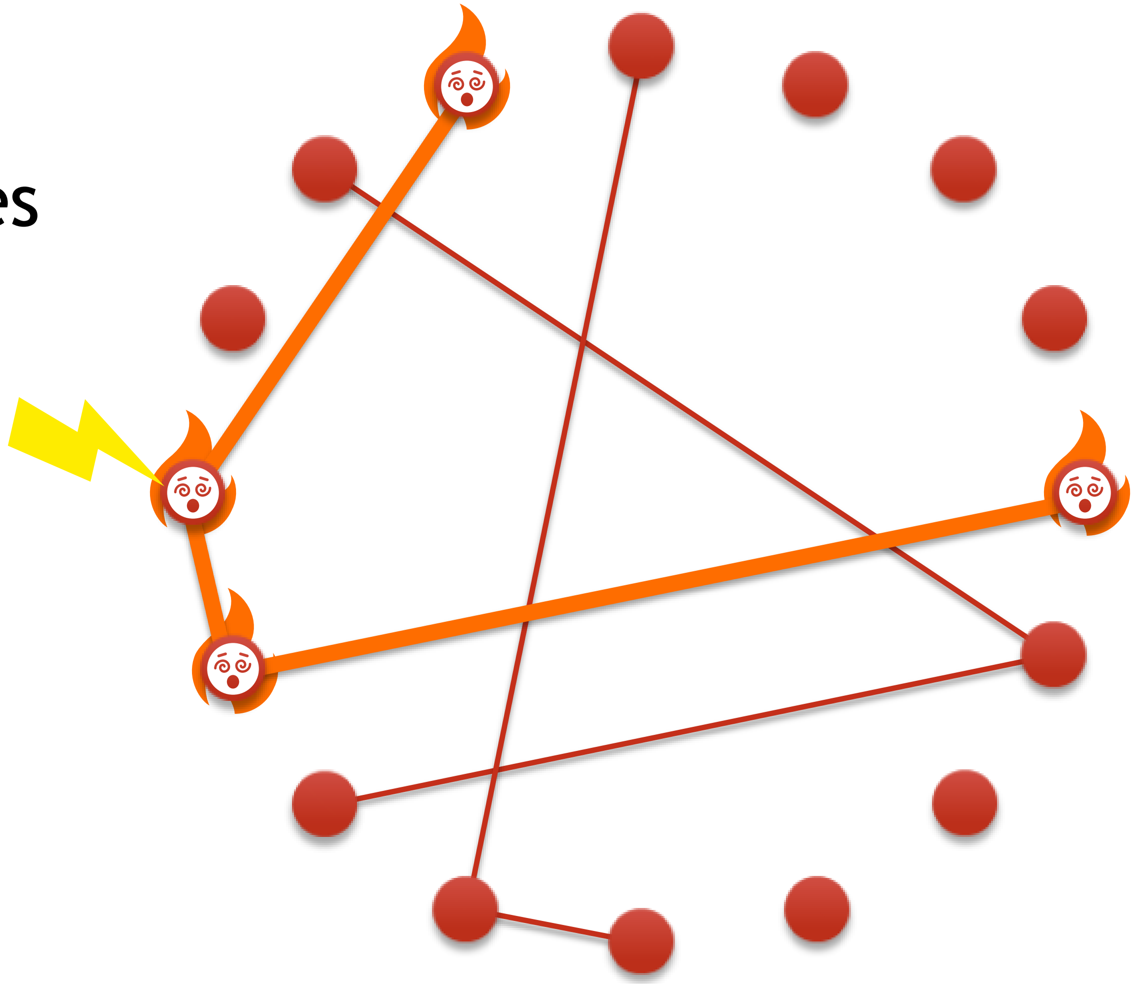
$$PPP\left(\frac{1}{n}\right)$$



The Model

Lightning Strikes

$$PPP(\lambda(n))$$

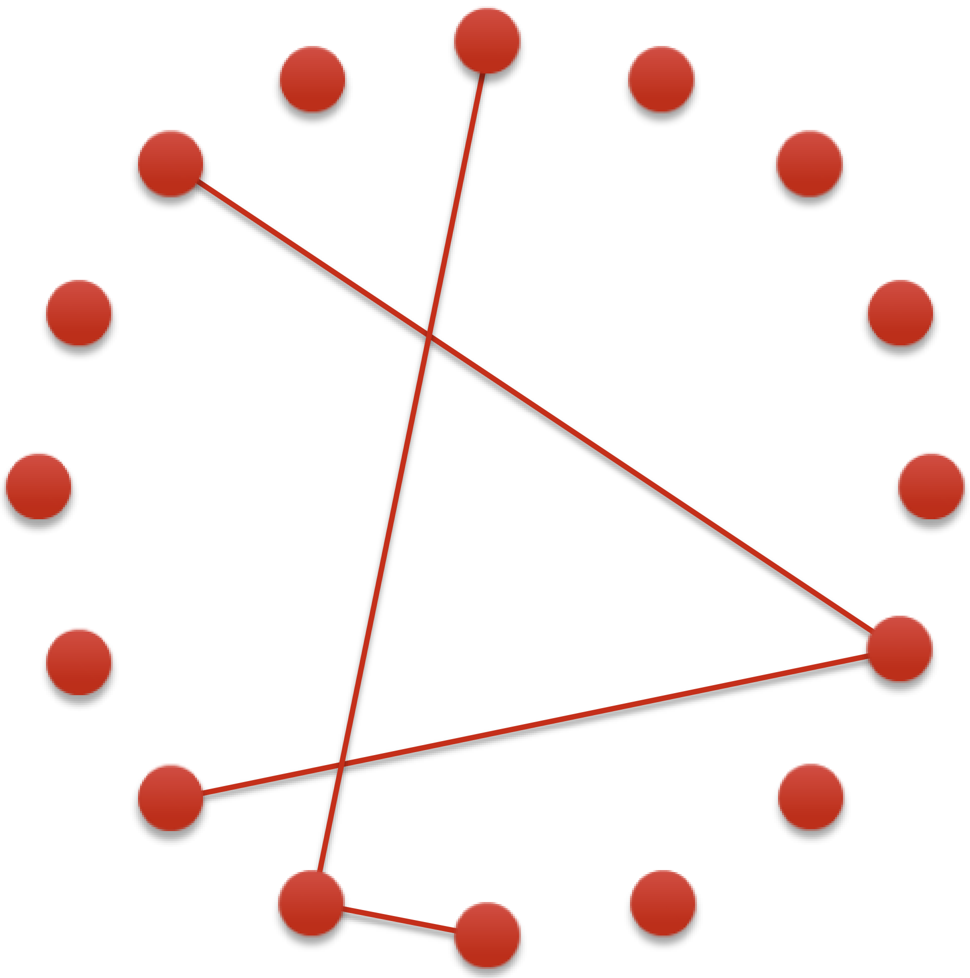


The fire removes all edges in the cluster containing the struck vertex.

The Model

Lightning Strikes

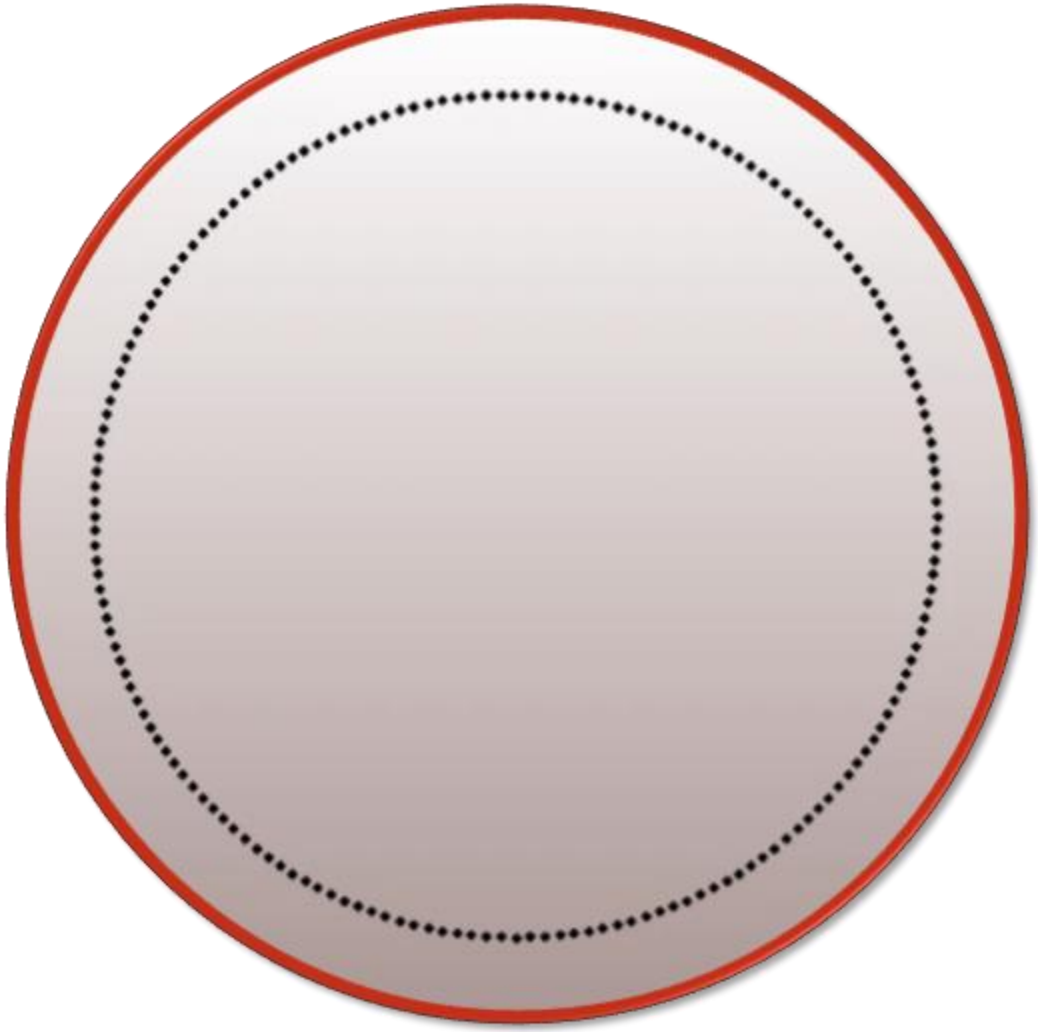
$$PPP\left(\lambda(n)\right)$$



The Model

$$\lambda(n) = n^{0.2}$$

Too much lightning for proper growth in the network.

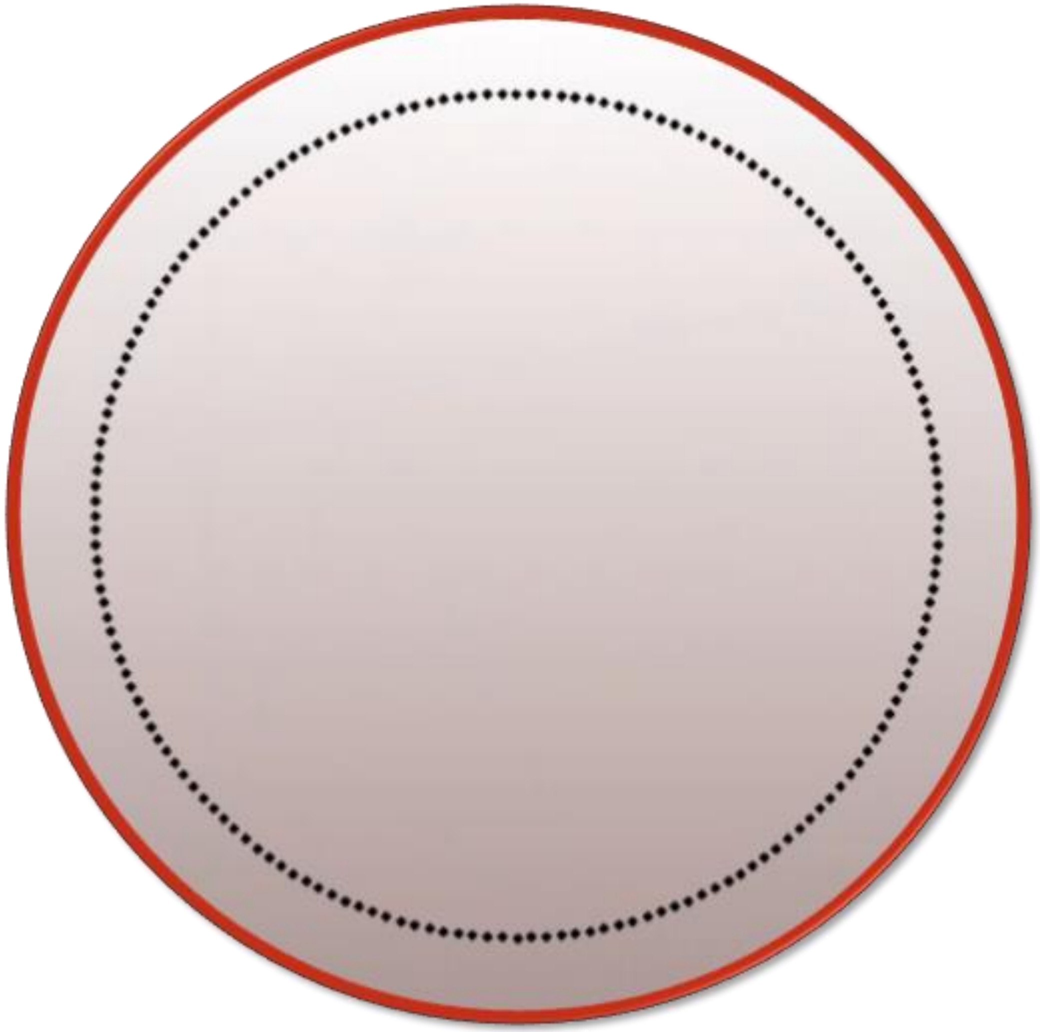


$$n = 180$$

The Model

$$\lambda(n) = n^{-1.2}$$

Too much growth in the network before a single strike burns a large proportion of the population.

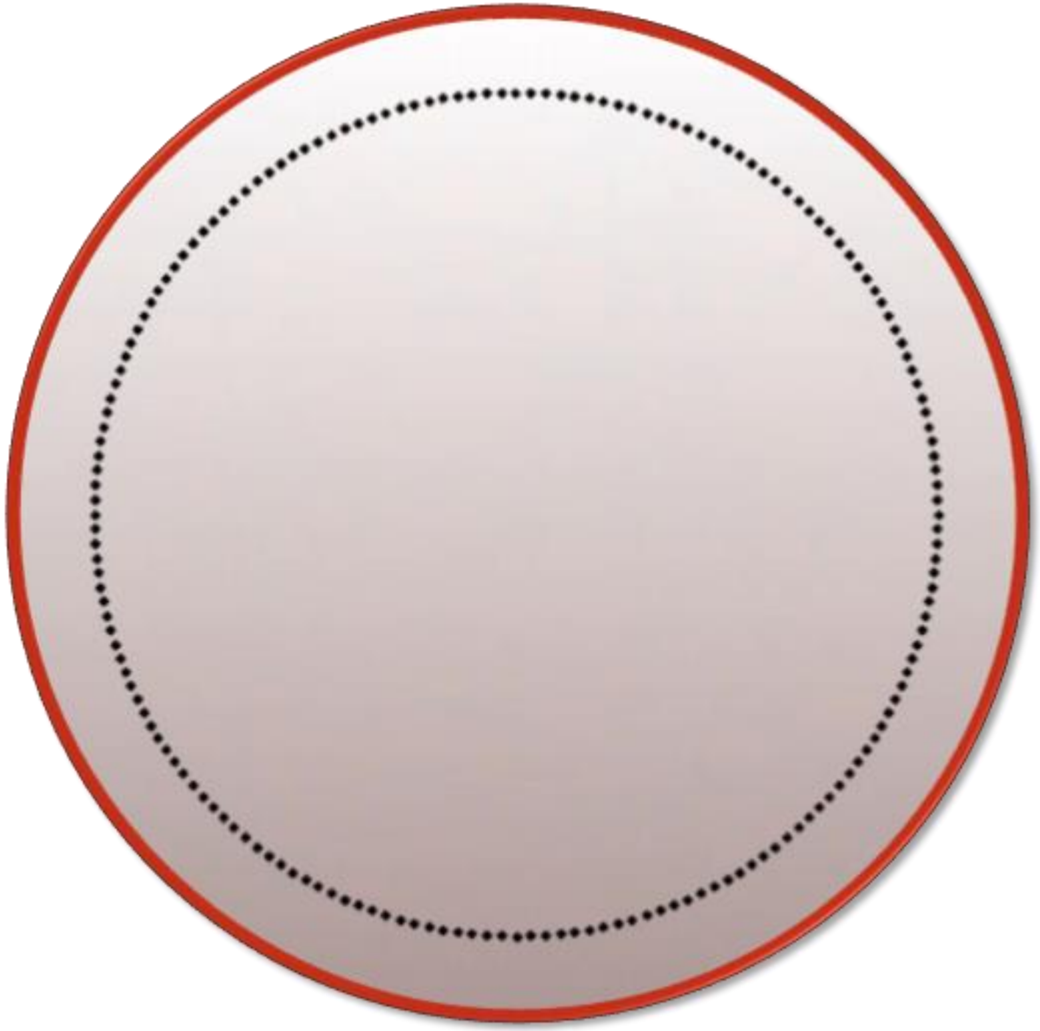


$$n = 180$$

The Model

$$\lambda(n) = n^{-0.5}$$

Nice balance between small occasional fires and growth in the model.



$$n = 180$$

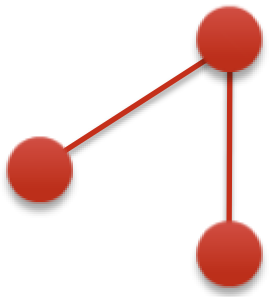
Criticality

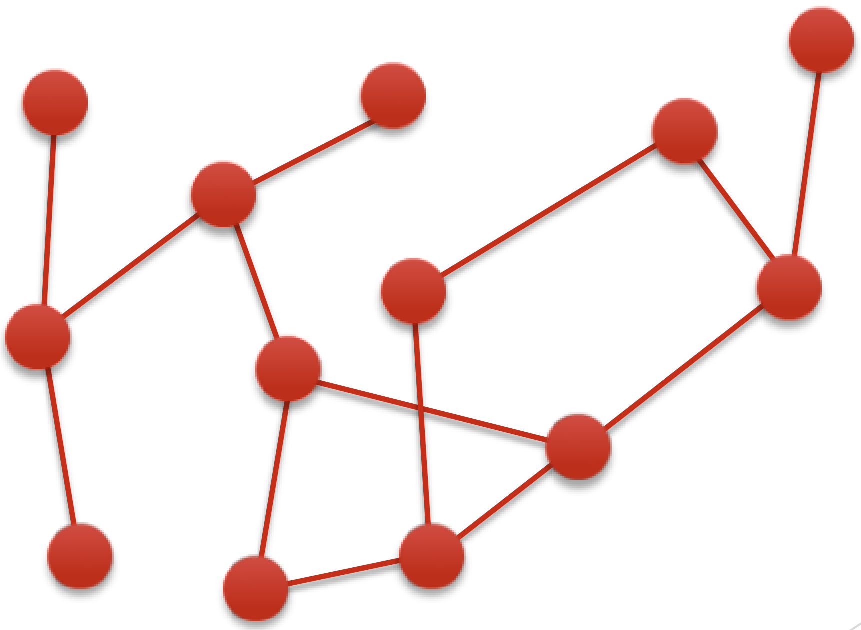
Criticality

The largest cluster is the quickest growing too.

Before anything, would you agree…

…that this gal… …grows slower than… …than this gal.





Criticality

Our plan is to look at quantities in this model as $n \to \infty$.

First, let's look at the size of the largest cluster:

$$\frac{|C^n_{max}(t)|}{n} \to 0$$

when $|C^n_{max}(t)| \ll n$.

Even the largest cluster in the n vertices will be of negligible size even when $n \to \infty$.

(Sub)critical

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Criticality

Our plan is to look at quantities in this model as $n \to \infty$.

First, let's look at the size of the largest cluster:

$$\frac{|C^n_{max}(t)|}{n} \to c > 0$$

when $|C^n_{max}(t)| \asymp n$.

Positive proportion of the n vertices will be burnt even when $n \to \infty$.

SUPERCRITICAL

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"GIANT BURNING CLUSTER"
=BAD

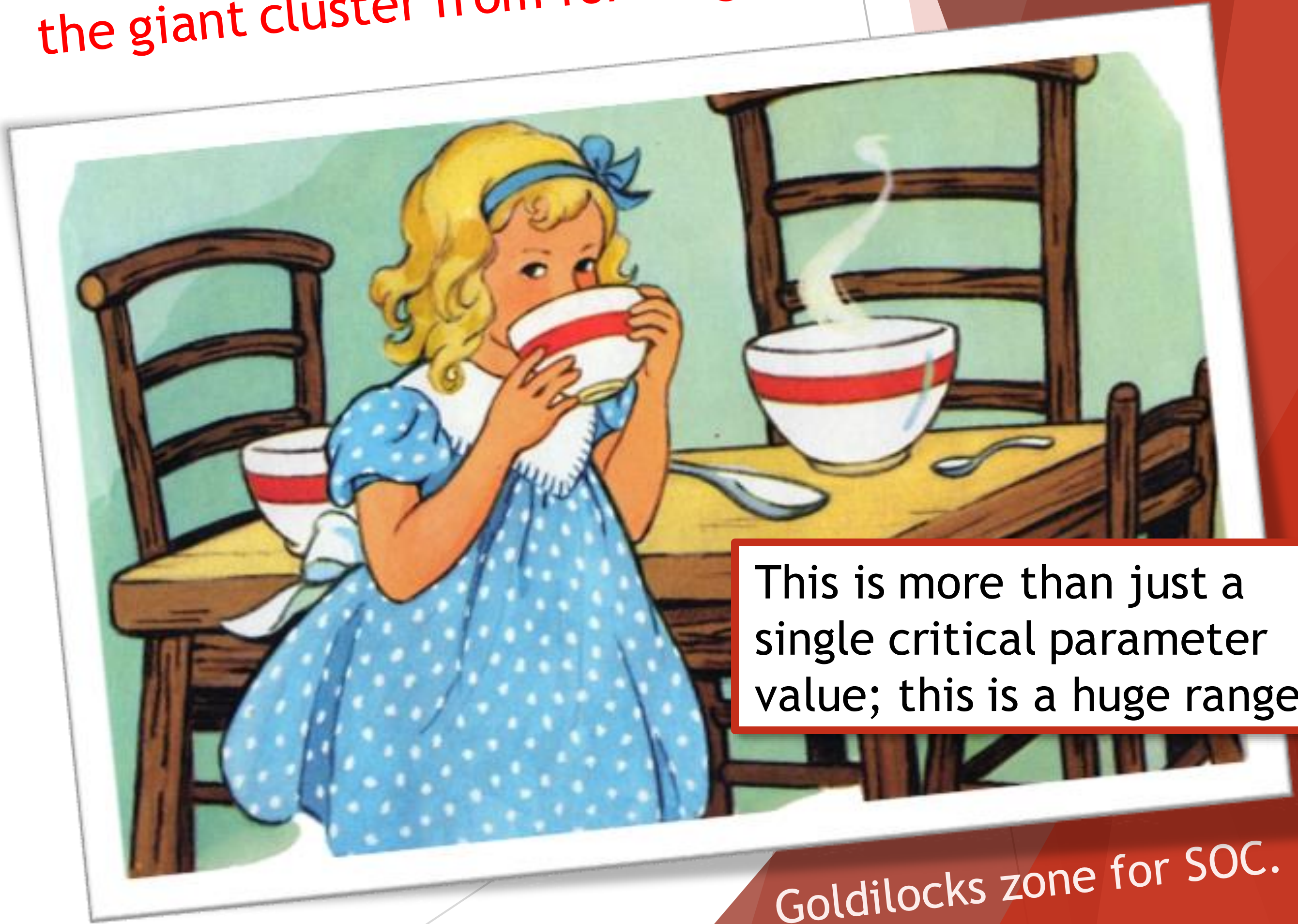
Self-organized Criticality

If we choose the lightning rate to behave like

$$n^{-1} \ll \lambda(n) \ll 1,$$

then the system sustains itself in a state of criticality.

The lightning is not too hot, not too cold, but just right to stop the giant cluster from forming.



This is more than just a single critical parameter value; this is a huge range!

Goldilocks zone for SOC.

Self-organized Criticality

But what even is “critical” behaviour?

As it turns out,…

…in the $n \to \infty$ limit, all clusters of any fixed size feel no effects of the lightning.

However,…

…the largest cluster is struck before it ever accumulates a mass comparable to n.

Critical Forest Fire Equations

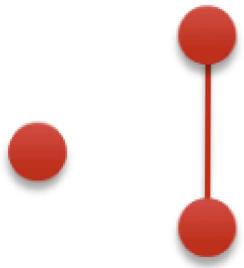
$v^n(t) \in \mathbb{R}^n$

The proportion of vertices in a cluster of size k.

$$v_k^n(t) := \frac{1}{n} \cdot |\{w = 1, \ldots, n : |Clus(w)| = k\}|$$

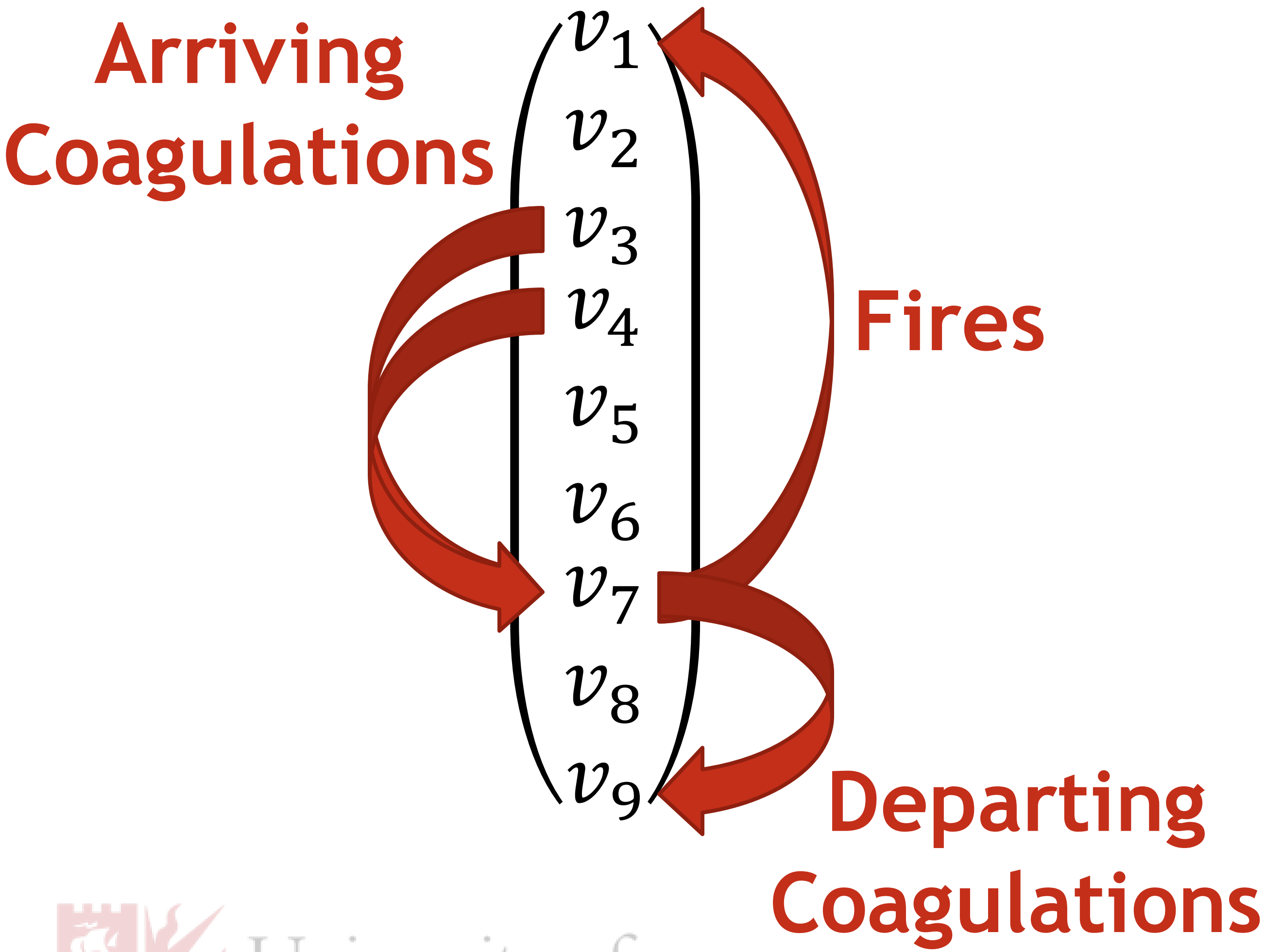
$$\sum_{k=1}^{n} v_k(t) = 1$$

Example: $n = 3$



$$v(0) = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix}$$

Critical Forest Fire Equations



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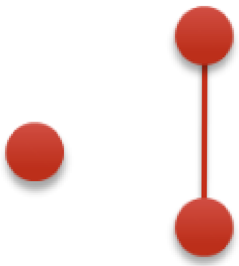

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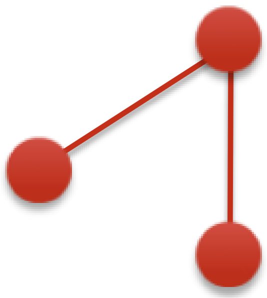


$$v(0.2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Critical Forest Fire Equations

Ráth and Tóth proved that $v_k^n \xrightarrow{\mathbb{P}} v_k$, pointwise for each $k \in \mathbb{N}$.

For suitable initial conditions, the cluster sizes evolve following these differential equations:

$$\dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k-1} \underbrace{v_l(t) v_{k-l}(t)}_{\text{Arriving Coagulations}} - \underbrace{k v_k(t)}_{\text{Departing Coagulations}}, \qquad k \geq 2$$

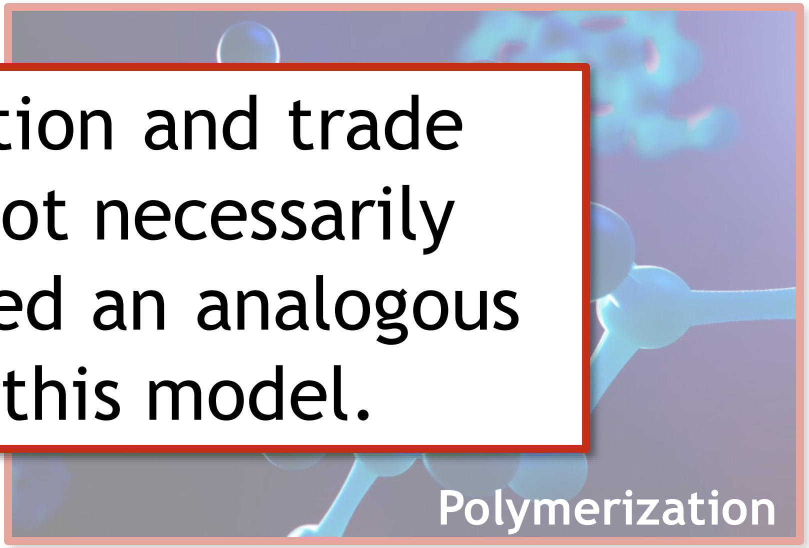
$$\sum_{k=1}^{\infty} v_k(t) = 1$$

Remark

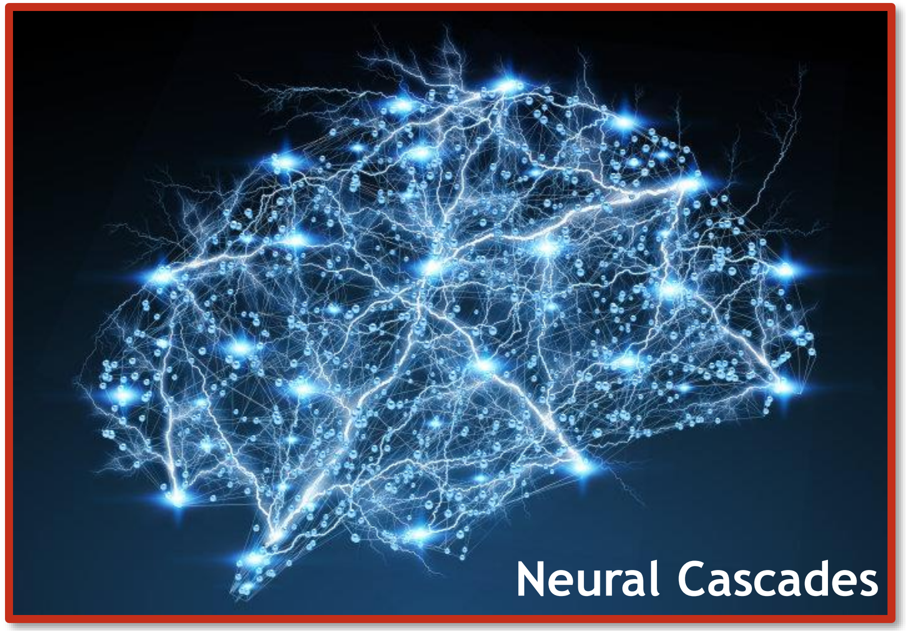
Many systems exhibiting self-organized criticality have a quantity which displays a power-law decay.

$$\sum_{k=R}^{\infty} v_k(t) \sim \frac{c(t)}{R^{1/2}}$$

Real-world Connections

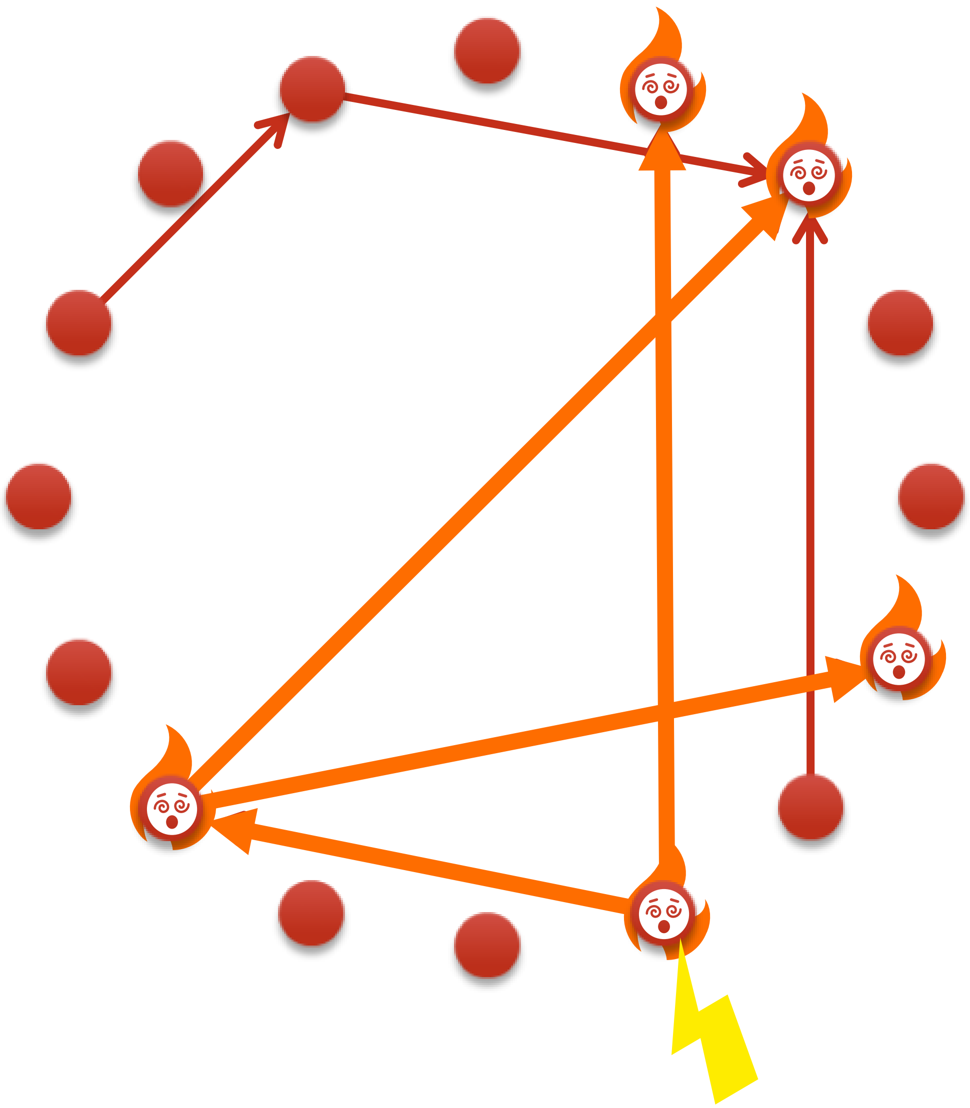


Neuron communication and trade dependencies are not necessarily reflexive, so we need an analogous directed variant of this model.



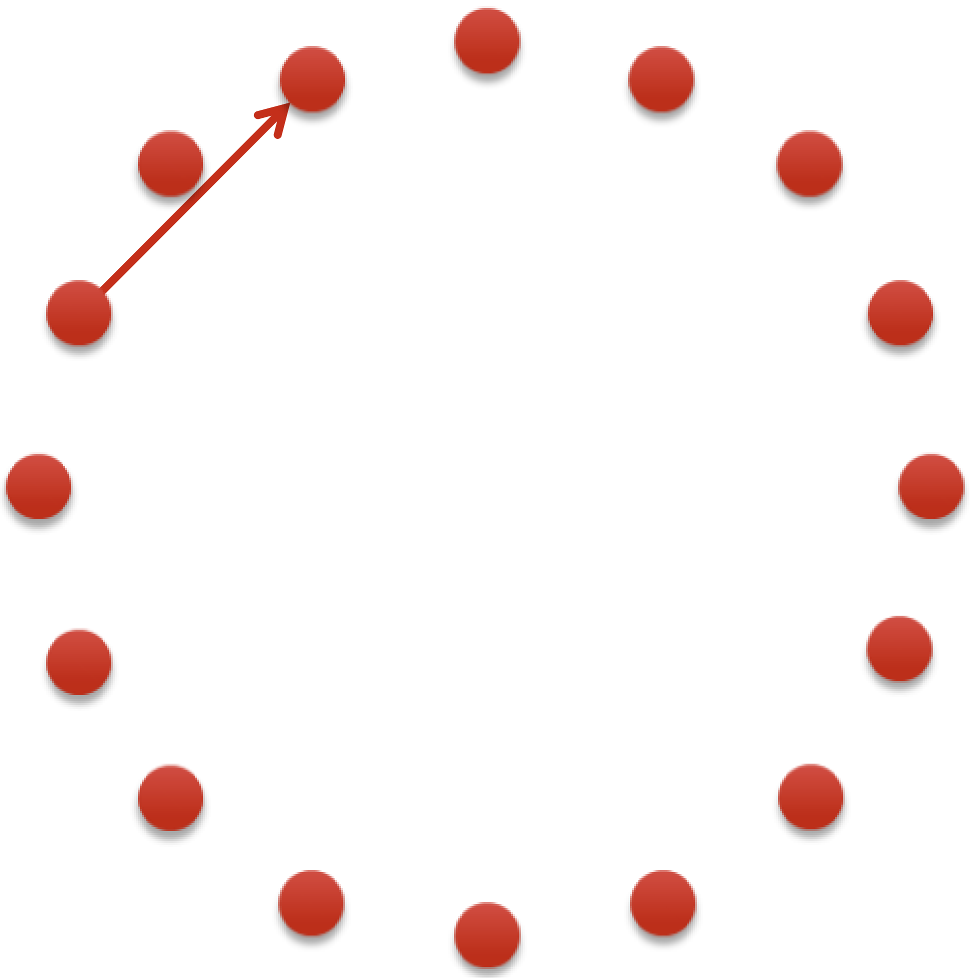
Directed variant

Now, fire can only propagate to vertices in the direction of the edges.



Directed variant

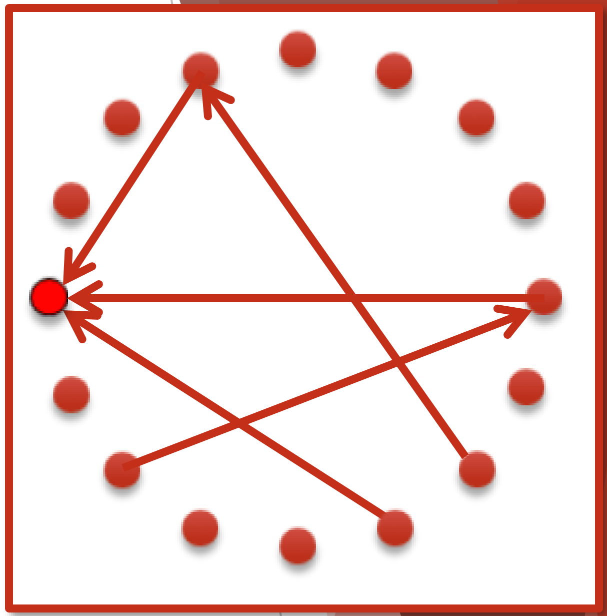
We remove all edges to and from a burning vertex.



Directed variant

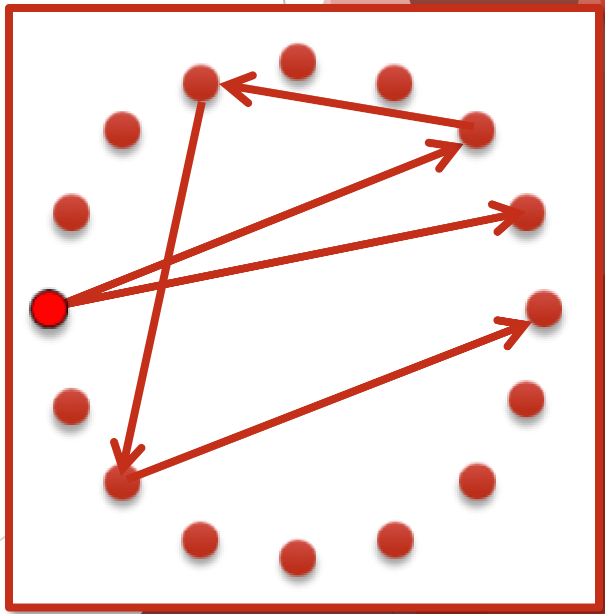
$\overleftarrow{v}^n(t) \in \mathbb{R}^n$ The proportion of vertices with an in-graph of size k.

$$\overleftarrow{v}^n_k(t) \coloneqq \frac{1}{n} \cdot |\{w = 1, \ldots, n : |In(w)| = k\}|$$



$\vec{v}^n(t) \in \mathbb{R}^n$ The proportion of vertices with an out-graph of size k.

$$\vec{v}^n_k(t) \coloneqq \frac{1}{n} \cdot |\{w = 1, \ldots, n : |Out(w)| = k\}|$$



Any Questions?

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